

Geometry & Topology

Workshop Turkey III

Harry Hwangil Baik (KAIST)

« Asymptotic translation length on the curve complexes
& normal subgroups of $\text{Mod}(S)$ ».

$G \curvearrowright (X, d)$ by isometries

For $x_0 \in X$, $g \in G$,

$$l_X(g) := \liminf_{n \rightarrow \infty} \frac{d(x_0, g^n x_0)}{n}$$

(asymptotic or stable translation length)

Note • By $\Delta \geq$, $l_X(g)$ does not depend on x_0 .

• $l_X(g^n) = n \cdot l_X(g)$.

• $l_X(g) \neq \tau_X(g) := \inf_{x_0 \in X} d(x_0, gx_0)$.
translation length.

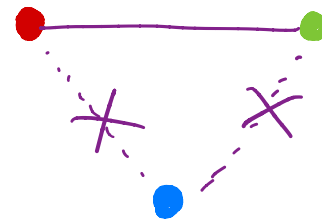
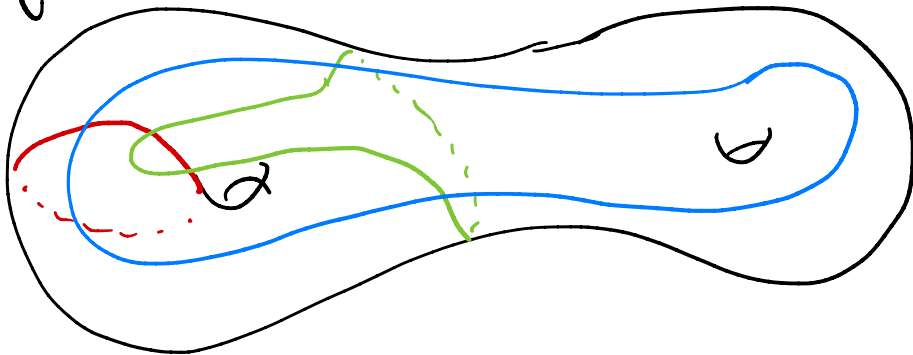
$G = \text{Mod}(S)$ mapping class group.

- S finite type surface.
- $\text{Mod}(S) = \text{Homeo}^+(S) / \text{isotopy}$.

$X = \begin{cases} \mathcal{T}(S) & \text{Teichmüller space w/ Teichmüller metric} \\ \mathcal{C}(S) & \text{curve graph w/ induced path-metric} \end{cases}$

- Vertices = isotopy classes of s.c.c.'s
- edges = disjoint representatives of two distinct vertices.

e.g.



Note. For $f \in \text{Mod}(S)$,

f is pseudo-Anosov


$$l_T(f) = l_T(f) > 0.$$

 (Masur - Minsky)

$$l_c(f) > 0.$$

© (Masur-Minsky + Gadre-Tsai)

f pseudo-Anosov. $\in \text{Mod}(S_{g,n})$

$$l_c(f) \geq 2gr + 30 \cdot |\chi(S_{g,n})| - 10 \cdot n$$

where $r = \#$ of real branches
of the invariant train-track τ
obtained by Bestvina-Handel
& $g \in \mathbb{N}$ s.t. $\exists$ some real
branch β of τ w/ $f^g(\beta) \supset \beta$.

© (Bowditch)

For any fixed S , $l_c(f)$ is a rational #

w/ uniformly bounded denominator.

Notation • For $f, g: \mathbb{N} \rightarrow \mathbb{R}_{>0}$,

We say $f \asymp g$ if $\exists C > 0$ s.t. $\frac{1}{C} < \frac{f(n)}{g(n)} < C, \forall n$.

• $H \leq \text{Mod}(S)$, $L_X(H) = \inf \{ l_X(f) \mid f \in H, f \neq 1 \}$.

Some known facts:

- (Penner) $L_T(\text{Mod}(S_g)) \asymp \frac{1}{g}$
- (Farb-Leininger-Margalit) $L_T(\mathcal{I}_g) \asymp 1$
- (Gradre - Tsai) $L_C(\text{Mod}(S_g)) \asymp \frac{1}{g^2}$
- (B.-Shin) $L_C(\mathcal{I}_g) \asymp \frac{1}{g}$
- (Kin-Shin) $L_C(B_n) \asymp \frac{1}{n^2}$
- (B.-Shin) $L_C(PB_n) \asymp \frac{1}{n}$.

Torelli subgroup
 $= \{ f \in \text{Mod}(S_g) \mid$
 $f \text{ acts trivially}$
 $\text{on } H_1(S_g) \}$

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Q1. Any relation
b/w l_T & l_C ?

Q2. l_X has
anything to do with
homological action?

About Q1. $\frac{l_T(f)}{l_c(f)}$ is unbounded from above.

but (Gradre-Hironaka-Kent-Leininger)

$$\inf_f \frac{l_T(f)}{l_c(f)} \sim \log(|X(S)|)$$

↑ up to uniform additive & multiplicative constants.

(Aougab-Taylor)

The ratio minimizers can be found

arbitrarily deep into the Johnson filtration.

About Q2 For $f \in \text{Mod}(S_g)$.

let $k(f)$ = the dimension of the subspace of $H_1(S_g; \mathbb{R})$
fixed by f .

$$\& L_\tau(k, g) := \min \{ L_\tau(f) \mid f \text{ p-A} \in \text{Mod}(S_g), k(f) \geq k \}.$$

$$\leadsto \begin{cases} L_\tau(0, g) = L_\tau(\text{Mod}(S_g)) \asymp \frac{1}{g} \quad (\text{Penner}) \end{cases}$$

$$\begin{cases} L_\tau(2g, g) = L_\tau(\tilde{I}_g) \asymp 1 \quad (\text{Farb-Leininger-Margalit}) \end{cases}$$

(Agol-Leininger-Margalit)

$$L_\tau(k, g) \asymp \frac{k+1}{g}$$

$$L_\tau(k, g)$$

$$\begin{array}{l} k=0. \quad \frac{1}{g^2} \\ k=2g \quad \frac{1}{g} \end{array}$$

$\sim \frac{k+1}{g^2} \quad ?!$

$$L_c(\text{Mod}(S_g)) \asymp \frac{1}{g^2}, \quad L_c(\mathcal{I}_g) \asymp \frac{1}{g}.$$

$$L_c(k, g) \asymp ??$$

$(B.-W_w)^* \equiv$ a uniform constant $C > 0$ s.t.

$$L_c(k, g) \geq \frac{C}{g(2g-k+1)}$$

- (Penner) $L_T(\text{Mod}(S_g)) \asymp \frac{1}{g}$
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Q3. Does L_C or L_T distinguish

proper normal subgroups from $\text{Mod}(S)$?

Said differently, can $l_C(f)$ or $l_T(f)$ tell if f is
a normal generator of $\text{Mod}(S)$ or not?

(Lanier-Margalit) $f \not\equiv p \cdot A$ in $\text{Mod}(S_g)$.

If $l_\tau(f) < \log \sqrt{2}$, then $\langle\langle f \rangle\rangle = \text{Mod}(S_g)$.

This means that $\exists C > 0$ s.t.

if $H \not\equiv \text{Mod}(S_g)$, then $L_\tau(H) \geq C$.

Conj $\exists C > 0$ s.t.

if $H \not\equiv \text{Mod}(S_g)$, then $L_C(H) \geq C/g$.

Equivalently, if $l_C(f) < C/g$, then $\langle\langle f \rangle\rangle = \text{Mod}(S_g)$.

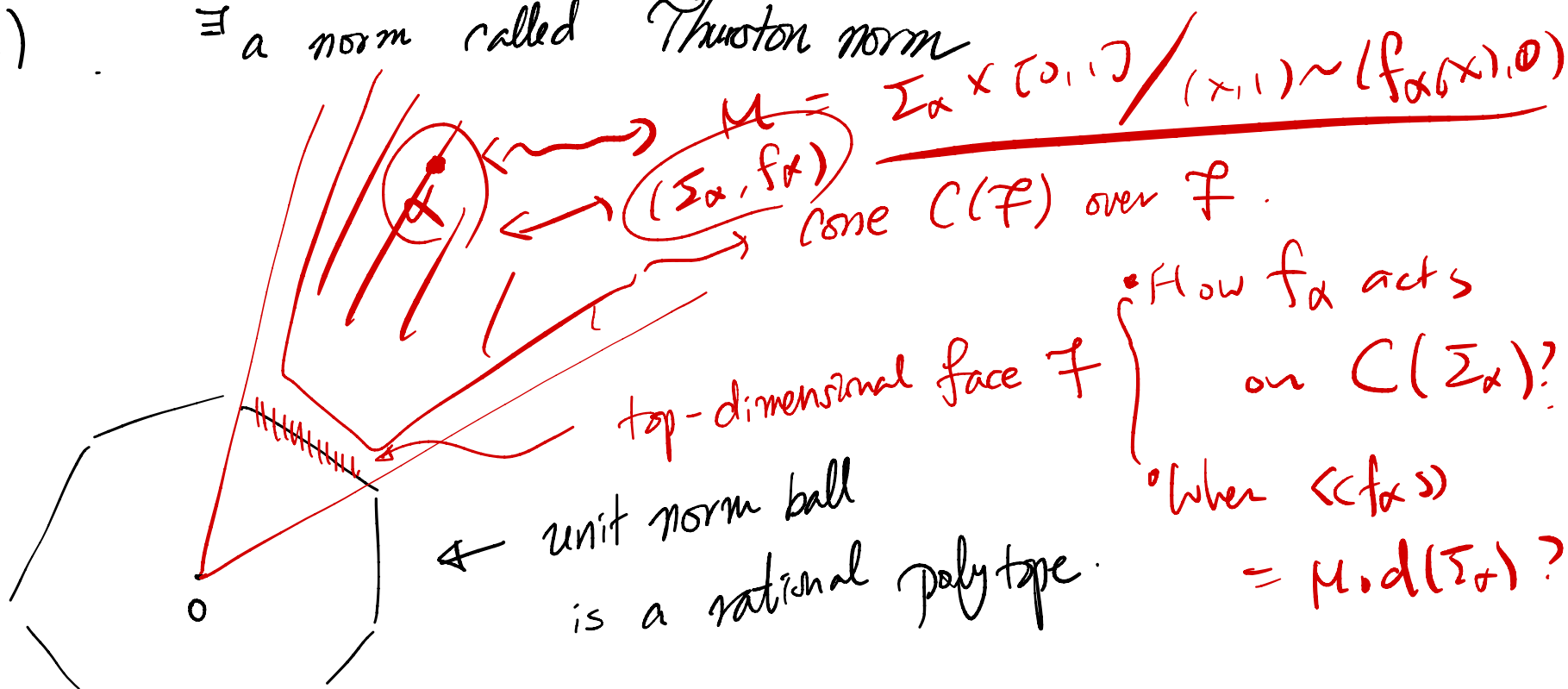
M^3 closed hyperbolic 3-manifold

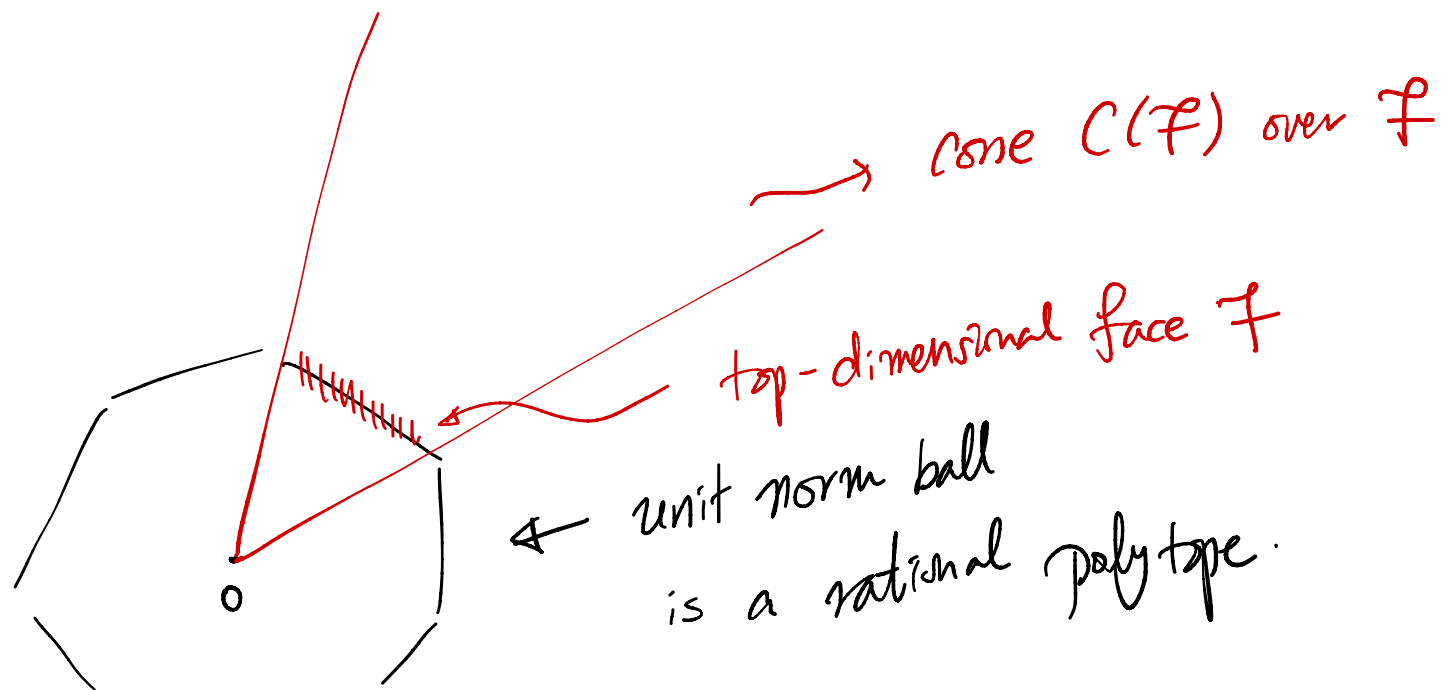
$$\begin{array}{ccc} \Sigma & \rightarrow & M^3 \\ \downarrow & & \\ S^1 & & \end{array} \quad \longleftrightarrow \quad M^3 \cong \Sigma \times [0,1] / (x,1) \sim (f(x),0), \quad \forall x =: M_f$$

for some $f \in \text{Homeo}(\Sigma)$.
monodromy

$H^1(M; \mathbb{R})$

$\cong$ a norm called Thurston norm





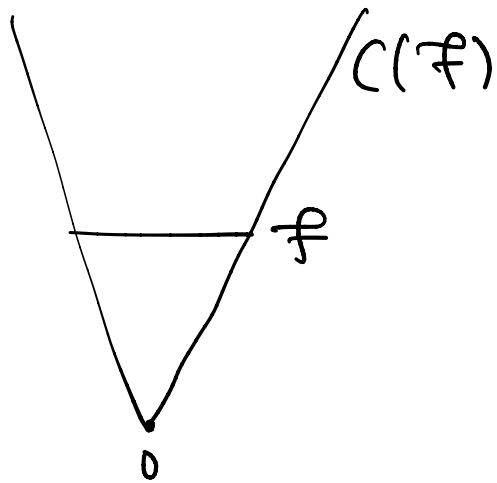
(Thurston) If a primitive integral class α in $C(F)$ corresponds to a fibration of M over S^1 ,

so does every other primitive integral class in $C(F)$.

In this case, F = a fibered face.

$C(F)$ = a fibered cone.

What matters to us :



For a fibered face $\mathcal{F}$ of M ,

each primitive integral class α in $C(\mathcal{F})$

expresses M as $S_\alpha \times [0,1] / (x,1) \sim (\psi_\alpha(x), 0)$.

$\alpha \longleftrightarrow (S_\alpha \text{ fiber}, \psi_\alpha \text{ monodromy})$

Thm (B. - Kim - Shin - Wu)

For a given fibered face $\mathcal{F}$ of M (closed fibered hyp 3-fold),

$\exists C > 0$ s.t $\forall$ primitive class α in $C(\mathcal{F})$,

$$l_{C(S_\alpha)}(\psi_\alpha) < C/g(S_\alpha).$$

Thm (B.-Kim-Shin-Wu)

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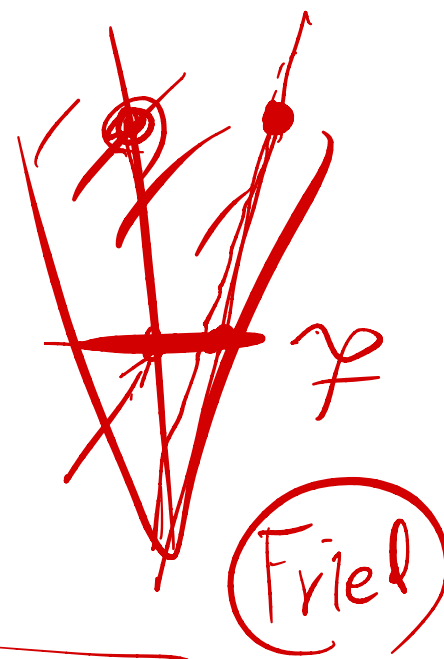
Conj $\exists C > 0$ s.t.

if $l_c(f) < C/g$, then $\langle\langle f \rangle\rangle = \text{Mod}(S_g)$.

What can we expect from here?

Thm (B. - Kim - Shin - Wu)

For a given fibered face $\mathcal{F}$ of M
 $\exists C > 0$ s.t. $\forall$ primitive class α in $C(\mathcal{F})$,
 $l_{C(S_\alpha)}(\psi_\alpha) < C/g(S_\alpha)$.



Conj $\exists C > 0$ s.t.

if $l_c(f) < C/g$, then $\langle\langle f \rangle\rangle = \text{Mod}(S_g)$.

What can we expect from here?

B. Strenner

Logically, nothing.

b/c C in Thm may be too big

compared to a desired C in Conj.

But morally, one might expect ...

Conj For a given fibered face F of M

for all but finitely many primitive integral classes α in $C(F)$,

$$\langle \psi_\alpha \rangle = \text{Mod}(S_\alpha).$$

But morally, one might expect ...

Conj For a given fibered face $\mathcal{F}$ of M

for all but finitely many primitive integral classes α in $C(\mathcal{F})$,

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this is necessary, since you can start with M_f

where $\langle\langle f \rangle\rangle \neq \text{Mod}(S)$.

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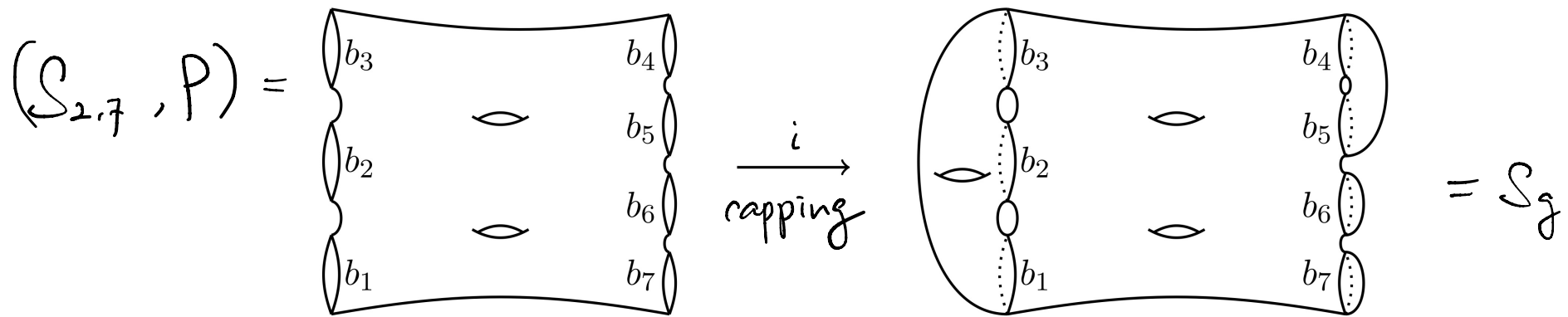
Thm (B. - Kim - Shin - Wu)

Conjecture holds for any 2-dim'l rational slice of $C(\mathcal{F})$.

6 Surfaces w/ boundary $S_{g,n}$ # of ∂ -components.

Label ∂ -components as $b_1, \dots, b_n$. Let P be a partition of ∂S .

e.g. $n=7$, $P = \{ \{b_1, b_2, b_3\}, \{b_4, b_5\}, \{b_6\}, \{b_7\} \}$.



$$\text{Mod}(S_{2,7}) \xrightarrow{i_*} \text{Mod}(S_g)$$

extending as identity on $S \setminus i(S_{2,7})$

$$\mathcal{I}(S_{2,7}, P) := i_*^{-1}(\mathcal{I}_g). \quad (\text{Putman})$$

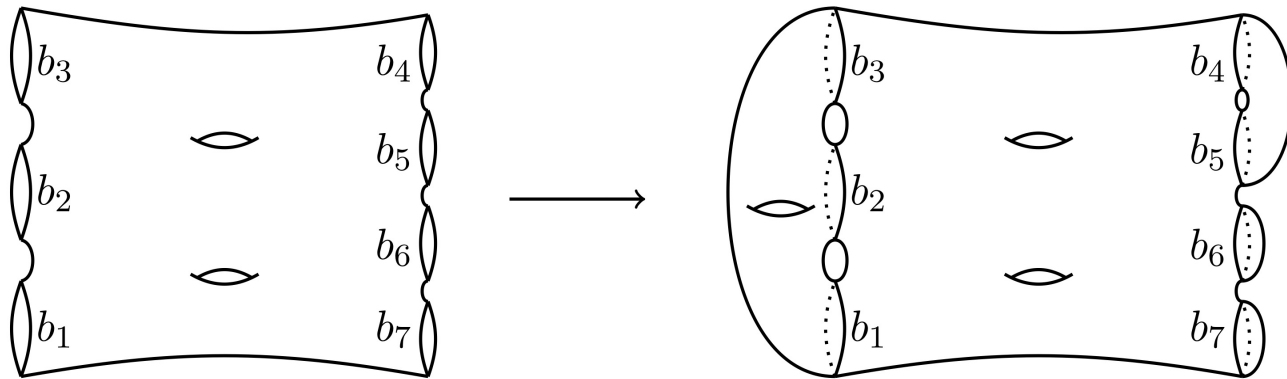
$$\text{Q} \quad L_c(\mathcal{I}(S_{g,n}, P)) \asymp ?$$

Thm (B. - Shin - Tranchida)

$$\frac{1}{|\chi(S_g^n)|} \lesssim L_C(\mathcal{I}(S_{g,n}, P)) \lesssim \frac{1}{|\chi(S_g^n)| - (n - |P|)}$$

where $|P|$ = the cardinality of the partition.

e.g. $P = \{ \{b_1, b_2, b_3\}, \{b_4, b_5\}, \{b_6\}, \{b_7\} \}$.



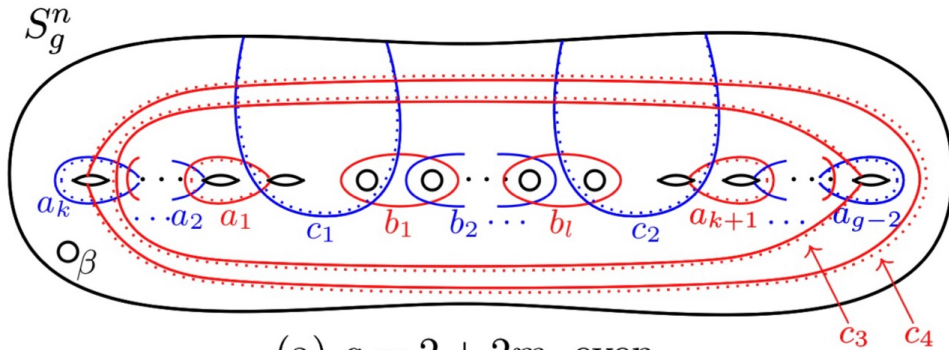
$$|P| = 4.$$

In particular, $L_C(\mathcal{I}(S_{g,n}, P_{\max})) \asymp \frac{1}{|\chi(S_g^n)|}.$

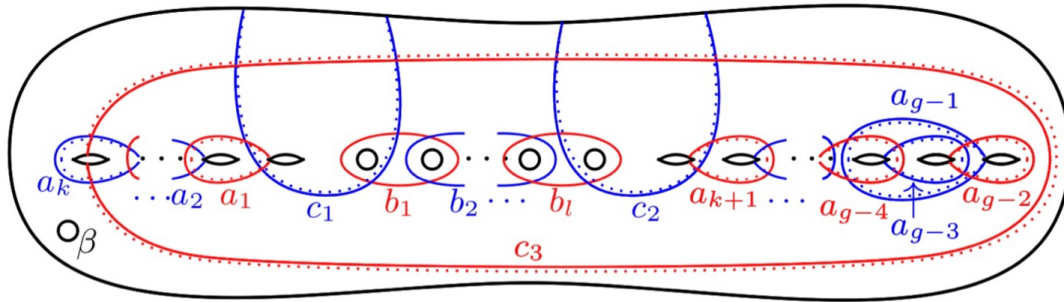
"Idea of the proof" (Lower bound — usual technique using Lefschetz #)

Suppose P is maximal, i.e.,

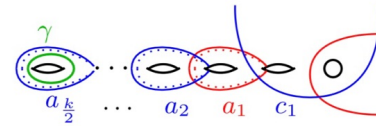
$$P = \{ \{b_1\}, \{b_2\}, \dots, \{b_n\} \}.$$



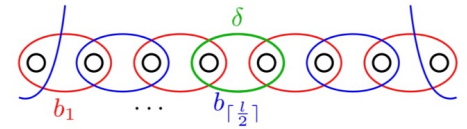
(a) $g = 2 + 2m$, even



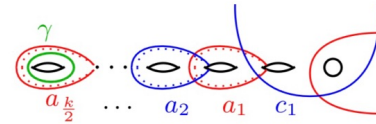
(b) $g = 3 + 2m$, odd



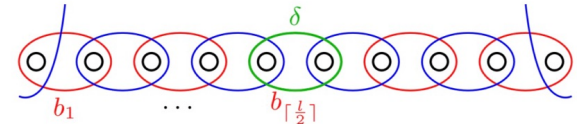
(a) When $k/2$ is even



(c) When $\lfloor n/2 \rfloor$ is even



(b) When $k/2$ is odd

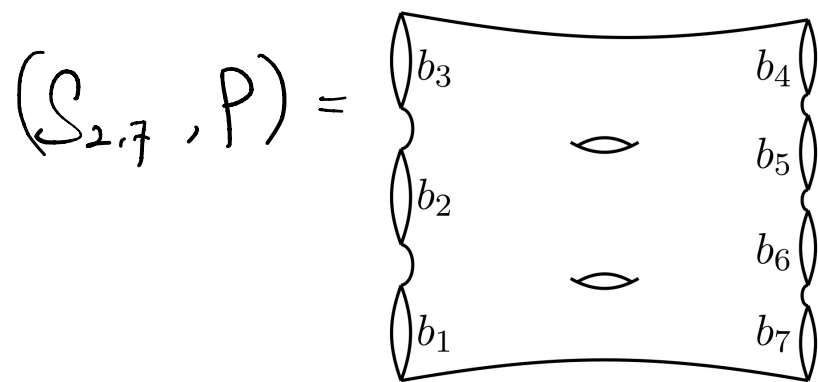


(d) When $\lfloor n/2 \rfloor$ is odd

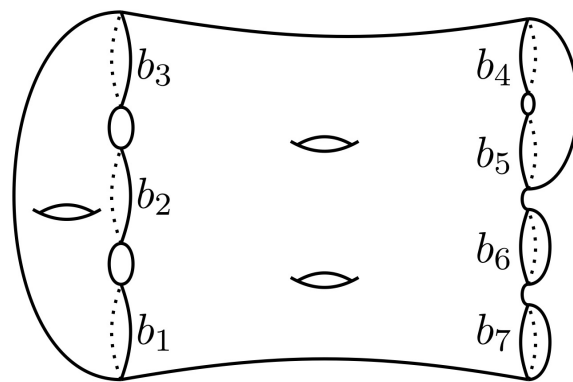
Roughly saying, γ is disjoint from both δ & $f^{|\chi(S)|}(\delta)$.

$$\text{So, } d(\delta, f^{|\chi(S)|}(\delta)) \leq 2 \Rightarrow l_c(f) \leq \frac{2}{|\chi(S)|}.$$

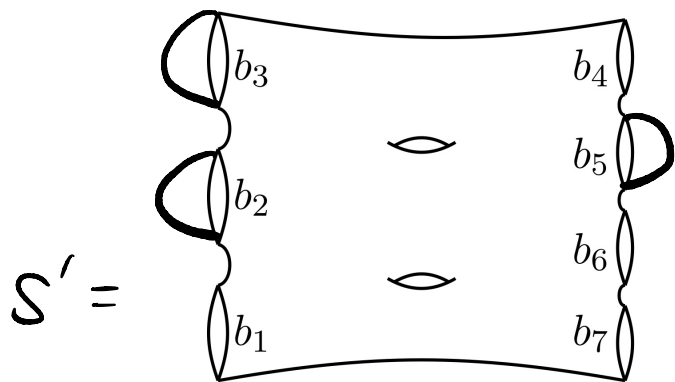
For a general P , let's see from an example.



$\xrightarrow[\text{capping}]{i}$



$\downarrow \text{Cap}$



$$P = \{ \{b_1, b_2, b_3\}, \{b_4, b_5\}, \{b_6\}, \{b_7\} \}$$

$$P' = \{ \{b_1\}, \{b_4\}, \{b_6\}, \{b_7\} \}$$

$\searrow$ maximal.

$\rightsquigarrow \quad \rho: \mathcal{I}(S_{2,7}, P) \rightarrow \mathcal{I}(S', P')$

$$\Rightarrow L_c(\mathcal{I}(S_{2,7}, P)) \leq L_c(\mathcal{I}(S', P')) \asymp \frac{1}{|\chi(S')|} = \frac{1}{|\chi(S)| - (n - |P|)}$$

Thank
you!