

Topics in Geometric Group Theory

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1 2018.06.25. : Mladen Bestvina

Definition 1.1. $\mathbb{F}_n$ is the free group of rank n .

Definition 1.2. The *outer automorphism group* of a group G is $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$.

History of geometric group theory:

1. Nielsen 1924, $\text{Out}(F_n)$: finitely presented.
2. Magnus 1935.
3. White 1936.
4. Stallings 1983, folding (Today's topic).
5. Culler–Vogtmann 1986, outer space.
6. Lipschitz metric (< 10 years old)
7. Train-tracks.

Definition 1.3. A *graph* is a CW-complex of dimension ≤ 1 .

Remark. To identify $\pi_1(\Gamma, v_0)$ with a free group, choose:

1. T maximum tree,
2. Orient and label all edges outside T .

Then $\pi_1(\Gamma, v_0) \simeq \mathbb{F}_n$ by “reading off the labels”.

Definition 1.4. A *morphism* $f : \Gamma \rightarrow \Gamma'$ between two graphs is a map that sends vertices to vertices and edges to edges. Here edges cannot collapse, and cannot map to edge-paths.

Definition 1.5. A morphism $f : \Gamma \rightarrow \Gamma'$ is an *immersion* if it is locally injective.

Example. Injective maps, covering maps, and composition of immersions.

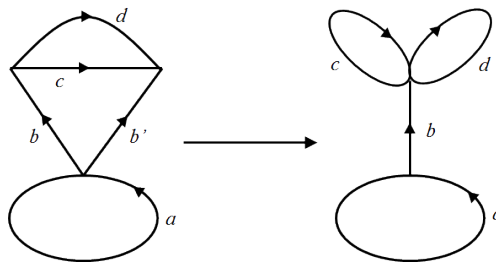


Figure 1: Example - not immersion.

Exercise. Immersions are injective in π_1 .

Definition 1.6. A *fold* is a morphism $f : \Gamma \rightarrow \Gamma / \sim$ obtained as follows: let e_1, e_2 be two oriented edges with the same initial vertex. Let $\Gamma' = \Gamma / e_1 \sim e_2$, then $f : \Gamma \rightarrow \Gamma'$ is a quotient map. There are four types of folds:

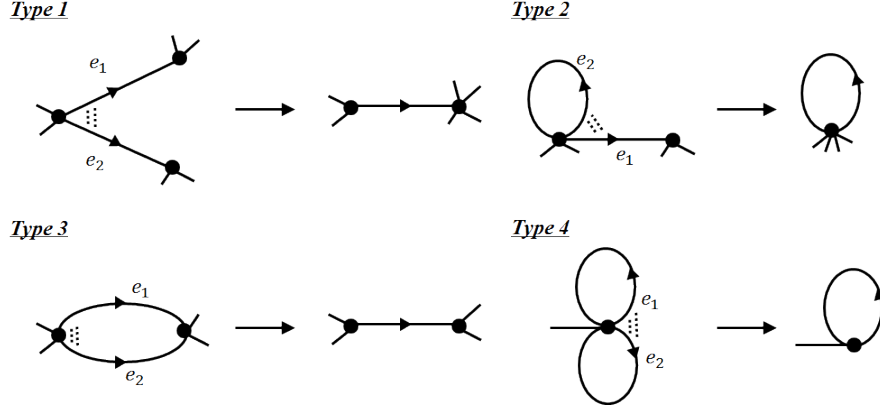


Figure 2: 4 Fold types.

Exercise. In π_1 , Type 1 & 2 are isomorphisms, and Type 3 & 4 are epimorphisms. [Hint: Type 1 - Choose maximum tree containing e_1, e_2 , Type 2 - subdivide e_1 .]

Theorem 1.7 (Stallings, 1983). *Every morphism $\Gamma \rightarrow \Gamma'$ between finite graphs can be factored as*

$$\Gamma = \Gamma_0 \rightarrow \Gamma_1 \rightarrow \cdots \rightarrow \Gamma_k \rightarrow \Gamma',$$

where the last map $\Gamma_k \rightarrow \Gamma'$ is an immersion and all the others are folds.

Example. Let $\mathbb{F}_2 = \langle a, b \rangle$, $H = \langle a^3b, \bar{a}bab, a^2\bar{b}a \rangle$.

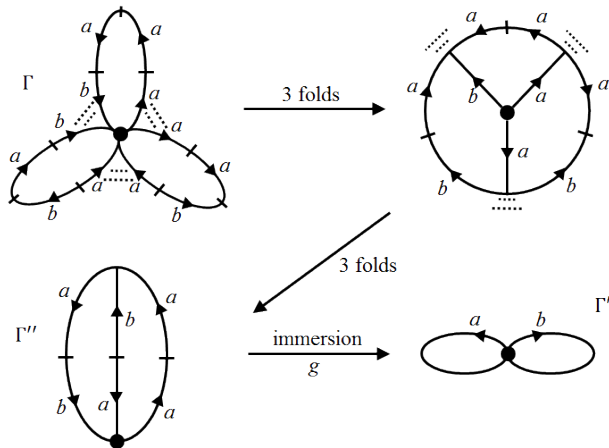


Figure 3: Morphisms $\Gamma \rightarrow \Gamma'$ corresponding to $H \rightarrow \mathbb{F}_2$.

Remark. 1. The image of $f_{\#} : \pi_1(\Gamma) \rightarrow \pi_1(\Gamma')$ is H . [$\text{img}_{\#} = H$, folds are epimorphism in π_1 .]

2. $g_{\#}$ is injective, so $\text{rank} H = 2$ and basis is $\{\bar{a}bab, a^2\bar{b}a\}$.

3. Does $a^2 \in H$? No. (If $a^2 \in H$, cycle is unique by a^2 is not cycle in H .)

Does $ab \in H$? No. Indeed, Γ'' completes to a covering space corresponding to H by adding labelled trees. Then ab is not a cycle in covering space.

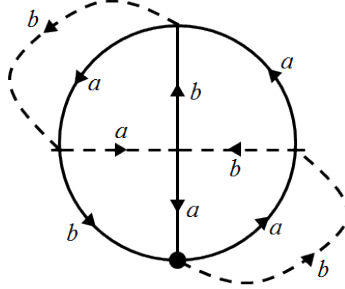


Figure 4: Γ'' completes to a covering space.

Note that such covering space exists by counting incoming edges and outgoing edges for each vertex. Therefore an immersion can be factored into covering map and monomorphism.

Theorem 1.8 (Marshall Hall's theorem). *If $H < \mathbb{F}_n$ is finitely generated, there is a finite index subgroup $H' < \mathbb{F}_n$ such that $H < H'$ and H is a free factor in H' .*

Proof. Complete the immersion g to a covering map by adding edges. Then any spanning tree in the original Γ'' extends to spanning tree in the covering space, using edges not contained in the original Γ'' . ■

Corollary 1.9. *Finitely generated subgroups of $\mathbb{F}_n$ are separable, i.e. if $H < \mathbb{F}_n$ is finitely generated, then for $g \in \mathbb{F}_n \setminus H$, there exists a finite index subgroup $H' < \mathbb{F}_n$ such that $H < H'$ and $g \notin H'$.*

Corollary 1.10. *$\mathbb{F}_n$ is Hopfian, i.e. every epimorphism $f : \mathbb{F}_n \rightarrow \mathbb{F}_n$ is an automorphism.*

Example. 1. $\text{BS}(2, 3) = \langle x, t \mid t^{-1}x^2t = x^3 \rangle$ is not Hopfian.

2. Let $\mathbb{F}_2 = \langle a, b \rangle$. Let $f : \mathbb{F}_2 \rightarrow \mathbb{F}_2$ given by $a \mapsto ab$ and $b \mapsto bab$. Since f is epimorphism, the last map g is actually a homeomorphism, so the rank of Γ_k in the Stallings' theorem is zero. If there are folds of type 3 or 4, then the rank decreases, so all folds are of type 1 or 2. Hence these are all isomorphism in π_1 and so is f .

Definition 1.11. A group is *residually finite* if

$$\bigcap \{H \mid H < G \text{ finite index}\} = 1.$$

Exercise. If a group G is finitely generated and residually finite, then G is Hopfian.

Theorem 1.12 (Nielsen, 1924). $\text{Aut}(\mathbb{F}_n)$ is finitely generated by the following automorphisms:

- (1) Permutations of the basis elements $\{a_i\}_{i=1}^n$,
- (2) Sign changes $a_i \mapsto a_i^{\pm 1}$,
- (3) (Nielsen automorphism) $a_1 \mapsto a_1 a_2$, other $a_i \mapsto a_i$.

The subgroup generated by automorphisms (1), (2) is called the *signed permutation group*.

Definition 1.13. Let Γ be a graph, T_1, T_2 the maximum tress in Γ . Labelling and orienting edges outside of T_1, T_2 give identifications, because label permutation induces automorphism (1), and opposite orientation induces automorphism (2). Then we have a commutative diagram:

$$\begin{array}{ccc} \pi_1(\Gamma, v_0) & \xrightarrow[\simeq]{\phi_{T_1}} & \mathbb{F}_n \\ \text{id} \parallel & & \downarrow \\ \pi_1(\Gamma, v_0) & \xrightarrow[\phi_{T_2}]{\simeq} & \mathbb{F}_n \end{array}$$

such map $\mathbb{F}_n \dashrightarrow \mathbb{F}_n$ is a *change of maximum tree automorphism*.

Example (Whitehead automorphisms). This is the case where T_1, T_2 have one edge.

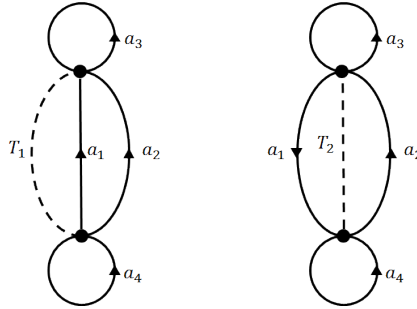


Figure 5: Example - Whitehead automorphism.

The map $f : \mathbb{F}_4 \rightarrow \mathbb{F}_4$ given by $a_1 \mapsto a_1$, $a_2 \mapsto a_2 a_1$, $a_3 \mapsto a_1^{-1} a_3 a_1$, and $a_4 \mapsto a_4$ is called a *Whitehead automorphism*.

Exercise. Whitehead automorphisms can be expressed as composition of (1)-(3).

Exercise. If T_1, T_2 have all edges in common except 1, the associated change of maximum tree automorphism is Whitehead.

Exercise. Let T, T' be two maximum trees of Γ . Then there is a sequence $T = T_0, T_1, \dots, T_m = T'$ of maximum trees so that T_i, T_{i+1} differ in only one edge.

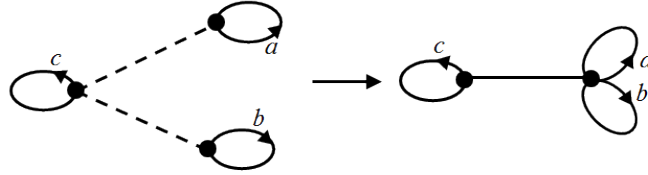
Corollary 1.14. *Every change of maximum tree automorphism is a finite composition of (1)-(3).*

Proof of Nielsen. Let $f \in \text{Aut}(\mathbb{F}_n)$. Consider a morphism of graphs $\Gamma \rightarrow R$ from subdivided rose Γ to the rose R , representing f . Also call it f . Factor it into folds:

$$\Gamma = \Gamma_0 \rightarrow \Gamma_1 \rightarrow \dots \rightarrow \Gamma_k \xrightarrow{\sim} R,$$

then the last map is homeomorphism, and all folds are type 1 & 2. We claim that for any of the folds $\Gamma_i \rightarrow \Gamma_{i+1}$, after choosing maximum trees $T_i < \Gamma_i$, $T_{i+1} < \Gamma_{i+1}$ carefully, the induced homomorphism in π_1 is generated by (1)-(3). In fact, it is Whitehead.

Type 1 : Identify



Type 2

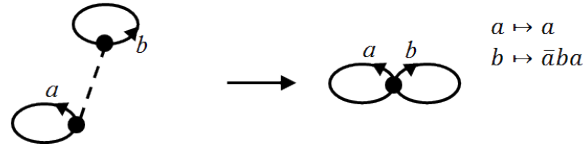


Figure 6: Maximum trees induce homomorphism.

Each Γ_i now has two maximum trees, one A_i corresponds to the fold $\Gamma_i \rightarrow \Gamma_{i+1}$, the other B_i corresponds to the fold $\Gamma_{i-1} \rightarrow \Gamma_i$. Then $(\Gamma_{i-1}, B_{i-1}) \rightarrow (\Gamma_i, A_i)$ and $(\Gamma_i, B_i) \rightarrow (\Gamma_{i+1}, A_{i+1})$, so the induced homomorphism is

$$\mathbb{F}_n \xrightarrow{\text{Wh}} \mathbb{F}_n - \text{ch} > \mathbb{F}_n \xrightarrow{\text{Wh}} \mathbb{F}_n$$

■

2 2018.06.26. : Mladen Bestvina

Definition 2.1. $G \curvearrowright X$ is *proper* if for any $K \subset X$ compact, $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite.

Definition 2.2. $G \curvearrowright X$ is *cocompact* if there exists $K \subset X$ compact such that $\bigcup_{g \in G} gK = X$.

There is a relation among:

- Outer space = space of graphs,
- Teichmüller space = space of hyperbolic metrics on a surface Σ ,
- symmetric space = space of Euclidean metrics on $\mathbb{T}^n$, i.e. $\mathrm{GL}_n(\mathbb{Z})$.

They are all contractible, all have proper action of the came up group.

Definition 2.3. Let Γ be a finite graph with no vertices of valence < 3 . A *metric* on Γ is a function $l : E(\Gamma) \rightarrow [0, \infty)$. A metric l is *positive* if $l > 0$, *non-degenerate* if the set of edges of length 0 forms a forest. The set of non-degenerate metrics on Γ is a cone (by scaling) i.e.

$$\Sigma(\Gamma) = \left\{ l : E(\Gamma) \rightarrow [0, \infty) \mid l \text{ non-degenerate, } \sum_e l(e) = 1 \right\},$$

is a simplex with some faces removed.

Definition 2.4. Let $R = \bigvee_1^n S^1$ be a rose, and Γ be a graph with no vertices of valence < 3 . A *marking* is a homotopy equivalence $h : R \rightarrow \Gamma$.

Caution. Usually, one defines a marking b defining h^{-1} , i.e. homotopy inverse.

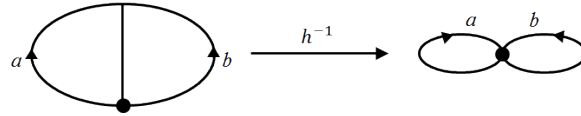


Figure 7: Example - marking.

Definition 2.5. Two markings $h_1, h_2 : R \rightarrow \Gamma$ is *equivalent* if they are homotopic.

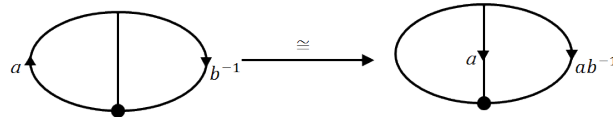


Figure 8: Example - equivalent marking.

Remark. A marking is *really* a choice of identification $\pi_1(\Gamma) = \mathbb{F}_n$.

Definition 2.6 (Culler–Vogtmann, 1986). The *Culler–Vogtmann outer space* $\mathcal{CV}_n$ is the set of equivalence classes of (Γ, l, h) , where Γ is a finite graph with no vertices of valence < 3 , l is a positive metric on Γ with total volume 1, $h : R \rightarrow \Gamma$ is a marking, and the equivalence relation is defined by

$$(\Gamma, l, h) \sim (\Gamma', l', h') \iff \begin{array}{ccc} R & \xrightarrow{h} & \Gamma \\ & \searrow h' & \downarrow \exists \varphi \text{ isometry} \\ & & \Gamma' \end{array}$$

Here the diagram is homotopy equivalent.

Definition 2.7. For any marked graph (Γ, h) we have the set of non-degenerate metrics $\Sigma(\Gamma, h) \subset \mathcal{CV}_n$. Each $\Sigma(\Gamma, h)$ has its topology. We define the *topology* on $\mathcal{CV}_n$ by the weak topology with respect to the collection

$$\{\Sigma(\Gamma, h) \mid \Gamma \text{ graph, } h : R \rightarrow \Gamma\},$$

i.e. a set $Z \subset \mathcal{CV}_n$ is closed iff $Z \cap \Sigma(\Gamma, h)$ is closed in $\Sigma(\Gamma, h)$ for each (Γ, h) .

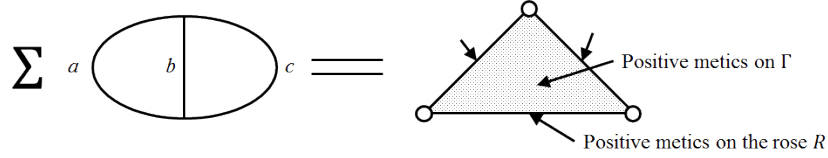


Figure 9: Example - cone 1.

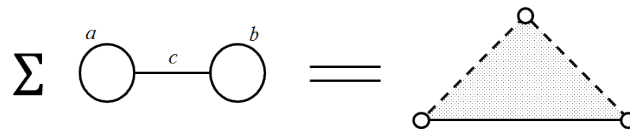


Figure 10: Example - cone 2.

Exercise. The intersection $\Sigma(\Gamma, h) \cap \Sigma(\Gamma', h')$ if nonempty, is a union of faces of each.

Exercise. (hard) In fact, it's a single face, so $\mathcal{CV}_n$ is a simplicial complex with a subcomplex removed.

Definition 2.8. $\text{Out}(\mathbb{F}_n) \simeq \{h : R \rightarrow R \mid h \text{ homotopy equivalence}\} / \text{homotopy}$ given by $[h] \mapsto h_* : \pi_1(R) \rightarrow \pi_1(R)$, so for $(\Gamma, l, h) \in \mathcal{CV}_n$ and $\Phi \in \text{Out}(\mathbb{F}_n)$, define (right) *group action* by $(\Gamma, l, h) \circ \Phi = (\Gamma, l, h\Phi)$.

$$\begin{array}{ccc} \Phi \circlearrowleft R & \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{h\Phi} \end{array} & \Gamma \end{array}$$

Example. Rank 2 picture: ¹

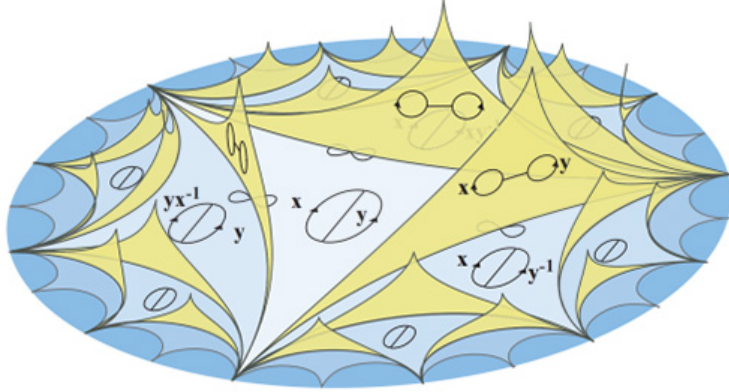


Figure 11: Example - Outer space $\mathcal{CV}_2$ in rank 2.

Proposition 2.9. *The stabilizer of (Γ, l, h) is isomorphic to the isometry group of (Γ, l) . In particular, it is finite.*

Proof. For $\Phi \in \text{Stab}(\Gamma, l, h)$,

$$\begin{array}{ccc} R & \xrightarrow{h} & \Gamma \\ & \searrow h\Phi & \downarrow \varphi \\ & & \Gamma \end{array} \quad \varphi h \simeq h\Phi, \quad \Phi \simeq h^{-1}\varphi h,$$

so we get $\text{Isom}(\Gamma, l) \rightarrow \text{Stab}(\Gamma, l, h)$, $\varphi \mapsto h^{-1}\varphi h$. By discussion, every isometry $\varphi : \Gamma \rightarrow \Gamma$ homotopic to identity is identity, so it is an isomorphism (for $n > 1$). ■

There are several sample questions about $\text{Out}(\mathbb{F}_n)$:

1. How big can a finite subgroup be?
2. How big can a free abelian subgroup be?
3. Are there any exotic subgroups? Tits alternative: any subgroup of $\text{Out}(\mathbb{F}_n)$ either contains a non-abelian free group or is virtually solvable.

¹<https://www.math.ku.dk/english/about/news/honorary-doctorate/>

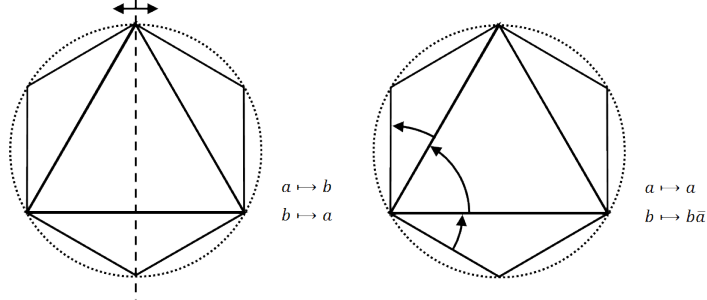


Figure 12: Example - automorphism of $\mathcal{CV}_2$.

Proposition 2.10. $\mathcal{CV}_n$ is path-connected. (in fact contractible)

Proof. Any point in $\mathcal{CV}_n$ can be connected by a path to a rose, we need to know that any rose Γ can be connected by a path to R . Fix a morphism $\Gamma \rightarrow R$ representing the inverse marking of Γ . Factor into folds as usual:

$$\Gamma = \Gamma_0 \rightarrow \Gamma_1 \rightarrow \cdots \rightarrow \Gamma_k \xrightarrow{\cong} R.$$

Every Γ_i can be viewed as a point in $\mathcal{CV}_n$ by removing hanging trees and valence ≤ 2 vertices. A fold $\Gamma_i \rightarrow \Gamma_{i+1}$ can be viewed as a path in $\mathcal{CV}_n$ from Γ_i to Γ_{i+1} . ■

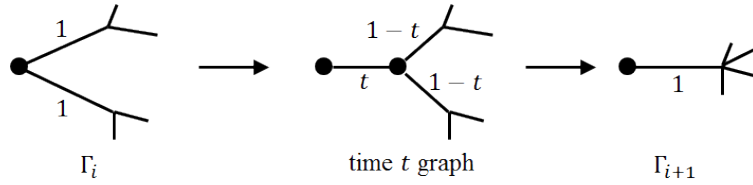


Figure 13: Folds can be viewed as a path.

Remark. There is an equivariant definition retract $K_n \subset \mathcal{CV}_n$ where the action is cocompact: let K_n be the union of simplices in the barycentric subdivision of each $\Sigma(\Gamma, h)$ that don't meet any missing faces.

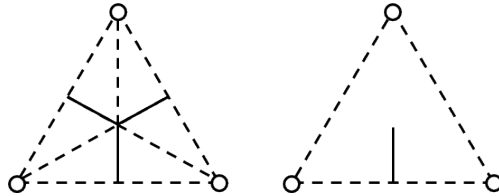


Figure 14: Example - construction of K_n .

Proposition 2.11. $\text{Out}(\mathbb{F}_n)$ is finitely presented.

Proof. We will need that $\ker[\text{Out}(\mathbb{F}_n) \rightarrow \text{GL}_n(\mathbb{Z}) \rightarrow \text{GL}_n(\mathbb{Z}/3)]$ is torsion-free. The strategy is to construct a free cocompact action of $\text{Out}(\mathbb{F}_n)$ on a 1-connected space X , then $\text{Out}(\mathbb{F}_n) = \pi_1(X/\text{Out}(\mathbb{F}_n))$ is finitely presented as $X/\text{Out}(\mathbb{F}_n)$ is a finite complex. Here K_n almost works, but there are finite stabilizers. To fix this, note that any finite group Q acts freely on a compact space $Q * Q * Q$, which is simply connected. Take $Q = \text{GL}_n(\mathbb{Z}/3)$. $\text{Out}(\mathbb{F}_n)$ acts on $X = K_n \times (Q * Q * Q)$ diagonally, on the second factor by projecting on Q . Done. ■

Definition 2.12. If G is a group, we let $\text{cd}(G)$ be the smallest integer k such that G acts properly and freely on a k -dimensional contractible space. $\text{cd}(G)$ is called the *cohomological dimension* of G .

Example. $\text{cd}(\mathbb{Z}) = 1$, $\text{cd}(\mathbb{Z}^n) = n$. As $\mathbb{Z}^n \curvearrowright X$, $X/\mathbb{Z}^n \simeq \mathbb{T}^n$, so $\dim X \geq n$.

Theorem 2.13. $\text{cd}(\ker(\text{Out}(\mathbb{F}_n) \rightarrow \text{GL}_n(\mathbb{Z}/3))) = 2n - 3$, and the largest rank of a free abelian subgroup in $\text{Out}(\mathbb{F}_n)$ is $2n - 3$.

Proof. We use the fact : $\text{cd}(\ker(\text{Out}(\mathbb{F}_n) \rightarrow \text{GL}_n(\mathbb{Z}/3))) \leq 2n - 3$. Observe that $\mathbb{Z}^{2n-3} < \text{Out}(\mathbb{F}_n)$: for example, if $n = 3$, $\{a \mapsto a, b \mapsto ba^k, c \mapsto a^m ca^l \mid k, m, l \in \mathbb{Z}\} \simeq \mathbb{Z}^3$, so $\mathbb{Z}^{2n-3} < \ker(\text{Out}(\mathbb{F}_n) \rightarrow \text{GL}_n(\mathbb{Z}/3))$. Hence $\text{cd}(\ker(\text{Out}(\mathbb{F}_n) \rightarrow \text{GL}_n(\mathbb{Z}/3))) = 2n - 3$. ■

Theorem 2.14 (Culler Zimmermann, ?, 1984?, Nielsen realization). *Every finite subgroup of $\text{Out}(\mathbb{F}_n)$ fixes a point of $\mathcal{CV}_n$.*

Theorem 2.15 (Stallings, 1960s). *If Γ is a group that contains $\mathbb{F}_n$ as a finite index subgroup, then Γ acts on a tree with all stabilizers finite.*

Proof of Nielsen realization. Let $H < \text{Out}(\mathbb{F}_n)$ be a finite subgroup. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{F}_n & \longrightarrow & \text{Aut}(\mathbb{F}_n) & \xrightarrow{p} & \text{Out}(\mathbb{F}_n) \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \mathbb{F}_n & \longrightarrow & p^{-1}H & \xrightarrow{p} & H \longrightarrow 1 \end{array}$$

By Stallings, $\mathbb{F}_n < p^{-1}H \curvearrowright T$ with finite stabilizers. Let $\mathbb{F}_n \curvearrowright T$ be free. Then $\Gamma = T/\mathbb{F}_n$ is a graph and $p^{-1}H/\mathbb{F}_n = H$ continues to act on Γ by isometries. ■

Exercise. Suppose $\varphi : \Gamma \rightarrow \Gamma$ is an isometry, and $\varphi_* : H_1(\Gamma, \mathbb{Z}/3) \rightarrow H_1(\Gamma, \mathbb{Z}/3)$ is an identity. Then φ is an identity.

3 2018.06.26. : Thomas Koberda

$\text{Homeo}^+(S^1)$ or $\text{Homeo}^+(I)$ is a group of orientation-preserving homeomorphisms.

Definition 3.1. A group G is (left) *orderable* if there exists a total order relation $\leq$ on G which is (left) invariant. $\leq$ is *bi-invariant* if it is left and right invariant.

Definition 3.2. G is *cyclically orderable* if $\exists w : G^3 \rightarrow \{0, \pm 1\}$ which is

- *left invariant*, i.e. $w(a, b, c) = w(ga, gb, gc)$,
- *cyclically invariant*, i.e. $w(a, b, c) = w(b, c, a)$,
- *anti-symmetric*, i.e. $w(a, b, c) = -w(b, a, c)$,
- *transitivity*, i.e. $w(a, b, c) = w(a, c, d) = 1 \Rightarrow w(a, b, d) = 1$,
- *total*, i.e. if a, b, c are distinct then $w(a, b, c) \neq 0$.

History of orderability:

1. Kokurin, Kopytov, L. Fuchs ~1940-60s. (primary algebraic interest)
2. Kaplansky's zero divisor conjecture: if G is a torsion-free group then $\mathbb{Z}[G]$ has no zero-divisors.
3. A. Rhemtulla, Botto–Mura.

Exercise. If G is left orderable, then G satisfies Kaplansky's conjecture.

Dynamical perspective: $\text{Homeo}^+(I)$, $\text{Homeo}^+(S^1)$.

Topology/Geometry perspective: Boyer–Garden–Watson.

Note (L -space conjecture). Let M^3 be 3-manifold. Then Heegaard–Floer homology of M corresponds one to one to orderability of $\pi_1(M)$.

Theorem 3.3 (Devoin, Navas, Ghys, Monod, Witte). *A countable group G is left orderable iff $G \hookrightarrow \text{Homeo}^+(I)$. Also, a countable group G is cyclically orderable iff $G \hookrightarrow \text{Homeo}^+(S^1)$.*

Proof. Let $f \in \text{Homeo}^+(\mathbb{R})$. Then f extends to the 2-point compactification of $\mathbb{R}$, i.e. $\mathbb{R} \cup \{\pm\infty\} \simeq I$. If $f \in \text{Homeo}^+(I)$ then $f \in \text{Homeo}^+((0, 1)) = \text{Homeo}^+(\mathbb{R})$. Suppose $G < \text{Homeo}^+(\mathbb{R})$. We claim that G admits a left-invariant order. Let $\mathbb{Q} = \{a_1, a_2, \dots\}$. We set $g < h$ iff $\min\{i \mid g(a_i) \neq h(a_i)\}$ satisfies $g(a_i) < h(a_i)$.

Now suppose G is countable group equipped with $\leq$. Write $G = \{g_0, g_1, \dots\}$. Define $t : G \rightarrow \mathbb{R}$ as follows: $t(g_0) = 0$, and suppose $\{t(g_0), \dots, t(g_i)\}$ has been defined. If $g_{i+1} > \max\{g_0, \dots, g_i\}$ (resp. $g_{i+1} < \min\{g_0, \dots, g_i\}$), then $t(g_{i+1}) = \max\{t(g_0), \dots, t(g_i)\} + 1$ (resp. $t(g_{i+1}) = \min\{t(g_0), \dots, t(g_i)\} - 1$). If $g_n < g_{i+1} < g_m$ with $n, m \leq i$, g_n maximum and g_m minimum, then $t(g_{i+1}) = \frac{t(g_n) + t(g_m)}{2}$. This gives an order preserving function $t : G \rightarrow \mathbb{R}$. Note that G acts on $t(G)$ faithfully, namely $g \cdot t(h) = t(gh)$, and this action is order preserving. ■

Exercise. Check the above relation $\leq$ defines a total left-invariant order on G .

Exercise. Check $G \curvearrowright t(G)$ extends continuously to $G \curvearrowright \overline{t(G)}$.

Exercise. Check $G \curvearrowright \overline{t(G)}$ extends continuously to $G \curvearrowright \mathbb{R}$.

Example. 1. $\mathbb{Z}$, more generally any subgroup of $(\mathbb{R}, +)$.

2. Finitely generated torsion-free nilpotent groups.

3. $H = \langle x, y, z \mid [x, y] = z, [z, x] = [z, y] = 1 \rangle$, *Heisenberg group*. Then

$$\mathbb{Z} \hookrightarrow H \xrightarrow{\phi} \mathbb{Z}^2$$

where ϕ is given by $x \mapsto x$, $y \mapsto x$, and $z \mapsto 1$. Fix order $\leq_{\mathbb{Z}^2}$ and $\leq_{\mathbb{Z}}$. First, if $\phi(g) <_{\mathbb{Z}^2} \phi(h)$ then $g < h$ in H . Second, if $\phi(g) = \phi(h)$, then $g^{-1}h \in \mathbb{Z}$, so if $g^{-1}h >_{\mathbb{Z}} 1$ then $g < h$ in H .

Exercise. Every left-order on $\mathbb{Z}^n$ comes from “half-space” construction.

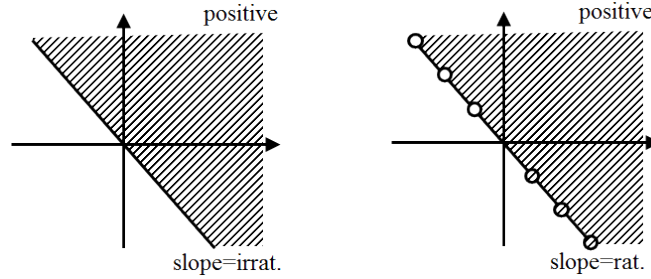


Figure 15: “Half spaces”.

Exercise. If $K \hookrightarrow G \twoheadrightarrow H$ and H, K are both left orderable, then G is left orderable.

Exercise. Check a finite cyclic group is not left orderable.

Definition 3.4. G is (finitely generated) *residually torsion-free nilpotent group* if $1 \neq g \in G$, there exists a torsion-free nilpotent group N and a homomorphism $\phi : G \rightarrow N$ such that $g \notin \ker \phi$.

Example. Free groups, surfaces, right-angled Artin groups, pure braid groups.

Definition 3.5. A group G is *metabelian* if $[G, G]$ is abelian.

Theorem 3.6 (Rhemtulla). *There is a finitely generated metabelian torsion-free non-left orderable groups.*

Remark. 1. All left orderable groups are cyclically orderable.

2. A finite cyclically orderable group is cyclic.

3. Mapping class groups are cyclically orderable.

4. (Baik-Samper theorem) If K is left orderable and H is cyclically orderable, and $K \hookrightarrow G \twoheadrightarrow H$, then G is cyclically orderable.

5. $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 \twoheadrightarrow \mathbb{Z}/2$, but $\mathbb{Z}/2 \times \mathbb{Z}/2$ is not cyclically orderable.

Remark. Let S be a *surface*, i.e. 2-dimensional real orientable manifold. Then S is characterized up to homeomorphism by *genus* g , number of *punctures* p , and number of ∂ *components* b . Then the *mapping class group* $\text{Mod}(S) = \pi_0(\text{Homeo}^+(S))(< \text{Aut}(\pi_1(S)))$. Say $S = S_{g,p,b}$ and $\text{Mod}(S) = \text{Mod}_{g,p,b}$. S is *hyperbolic type* surface if $3g - 3 + p + b > 0$. $\tilde{S}$ is the *universal cover* identified with (convex subset of) $\mathbb{H}^2$.

Theorem 3.7 (Nielsen, Handel–Thurston). *Say $S_{g,1} = S_{g,1,0}$. Then $\text{Mod}_{g,1}$ is cyclically orderable, and $\text{Mod}_{g,0,1}$ is left orderable.*

Proof. (Sketch) For $\psi \in \text{Mod}(S) = \text{Mod}_{g,1}$, let $\Psi \in \text{Homeo}^+(S)$ be a representation of ψ . Then $\Psi \curvearrowright S \leftarrow \mathbb{H}^2 \curvearrowright \tilde{\Psi}$, so $\tilde{\Psi}$ sends geodesics to quasi-geodesics. As $\tilde{\Psi}$ sends end-points to end-points, we get a continuous map $\partial\tilde{\Psi} \curvearrowright \partial\mathbb{H}^2 = S^1$. As $\partial\tilde{\Psi}$ is independent of $\tilde{\Psi}$, so we can write $\partial\psi$. By fixing lift $\tilde{p}$ of p , eliminate the ambiguity of a choice of lift. We can say $\partial\psi$.

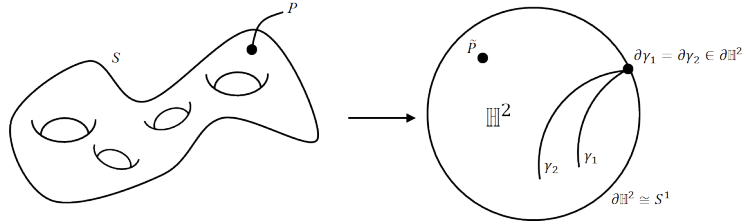


Figure 16: Surface with $g, p = 1$ and $b = 0$.

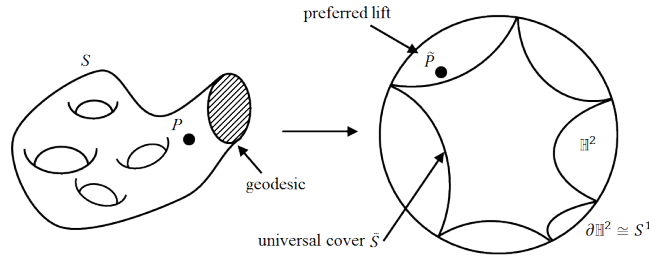


Figure 17: Surface with $g, p = 0$ and $b = 1$.

Then $\psi \mapsto \partial\psi$ is a homomorphism $\text{Mod}_{g,1} \rightarrow \text{Homeo}^+(S^1)$. This map is faithful. We can similarly prove for $\text{Mod}_{g,0,1}$. ■

Our basic theme is “dynamical interpretation of analysis”.

Definition 3.8. For $f \in \text{Homeo}^+(M)$, $\text{supp}(f) = \{x \in M \mid f(x) \neq x\}$.

Note. When can two homeomorphisms (diffeomorphisms) commute? For example, $f(x) = x + 1$ and for any $h \in \text{Homeo}^+((0, 1))$,

$$g(x) = \begin{cases} x, & \text{if } x \in \mathbb{Z}, \\ n + h(t), & \text{if } x = n + t, t \in (0, 1). \end{cases}$$

Theorem 3.9 (Kopell’s lemma). *Let $f, g \in \text{Diff}^2([0, 1])$. Suppose $[f, g] = 1$. Suppose further that $\text{Fix}(f) \cap (0, 1) = \emptyset$ and $\text{Fix}(g) \cap (0, 1) \neq \emptyset$, where $\text{Fix}(f) = [0, 1] \setminus \text{supp}(f)$. Then g is the identity.*

Caution. Actually Lipschitz condition on f' is enough. But $f, g \in \text{Diff}^1([0, 1])$ is *not enough*: there is a counter-example.

Corollary 3.10. *Let $f, g \in \text{Diff}^2([0, 1])$ with $[f, g] = 1$. Let A, B be components of $\text{supp}(f)$, $\text{supp}(g)$, respectively. Then either $A \cap B = \emptyset$ or $A = B$.*

Proof of Kopell’s lemma. Choose a homeomorphism $h : (0, 1) \rightarrow \mathbb{R}$ and a topological conjugacy from f to $x \mapsto x + 1$. Let $J = (h^{-1}(0), h^{-1}(1))$ be the fundamental domain for f . Then $f^n(J) \cap J = \emptyset$ if $n \neq 0$ and $I \setminus \bigcup f^n(J)$ is countable. As f is C^2 , $\log |f'|$ is Lipschitz. For $a, b \in J$,

$$\sum_{i=-\infty}^{\infty} |\log |f'(f^i(a))| - \log |f'(f^i(b))|| \leq K_1,$$

for some $K_1 > 0$ independent of a, b . Also, there exists $K_2 > 0$ such that

$$\frac{1}{K_2} \leq \left| \frac{(f^n)'(a)}{(f^n)'(b)} \right| \leq K_2.$$

Set $\phi = g^m$, then $[f, \phi] = 1$. Then for all n ,

$$\phi(f^n(x)) = f^n(\phi(x)), \quad \frac{(f^n)'(x)}{(f^n)'(\phi(x))} = \frac{\phi'(x)}{\phi'(f^n(x))}.$$

Suppose $g \neq \text{id}$. Then g has a fixed point p such that $g^k(x) \rightarrow p$ for x near p , $k \rightarrow \infty$. Then $|\frac{(f^n)'(x)}{(f^n)'(\phi(x))}|$ is bounded from 0 if x is close enough to p . As p is fixed, $\forall \epsilon > 0 \exists m > 0$ such that $|\frac{g^m(x) - p}{x - p}| < \epsilon$, x near p . In other words, $\forall \delta > 0 \exists m > 0$ such that $|\phi'(x)| = |(g^m)'(x)| < \delta$. As 0 is an accumulation point of $\text{Fix}(g)$, $\phi'(0) = 1$. Then $\forall \delta > 0 \exists n > 0$ such that $|\phi'(f^n(x))| \geq 1 - \delta$. So $|\frac{\phi'(x)}{\phi'(f^n(x))}| \leq \frac{\delta}{1 - \delta}$, contradiction. ■

Exercise. Check all details in the proof.

Theorem 3.11 (Plante–Thurston). *If $N < \text{Diff}^2(M)$ is nilpotent, then N is abelian.*

Theorem 3.12 (Hölder). *If $G \curvearrowright \mathbb{R}$ (or S^1) freely, then G is abelian.*

Proof of Plante–Thurston for $M = I$. Note $N \curvearrowright I$. Assume N has no global fixed points. Let $1 \neq g \in Z(N)$. Take $1 \neq f \in N$ with a fixed point. Let $x \in \partial\text{Fix}(f)$, $f(x) = x$. We claim that $g(x) = x$. If not, there exists $[a, b]$, $a \neq b$ such that $\{a, b\} = \{\lim_{n \rightarrow \pm\infty} g^n(x)\}$. Since g, f commutes, f fixes a and b . By Kopell’s lemma, if $g(x) \neq x$ then $f = 1$ on $[a, b]$. This violates $x \in \partial\text{Fix}(f)$. Thus $g(x) = x$. Now, let $y \in \text{Fix}(g) \cap (0, 1)$. Since $g \in Z(N)$, $h(y) = y$ for all $h \in N$, i.e. y is a global fixed point of N . Contradiction. ■

Exercise. Check we may assume $x \in \partial\text{Fix}(f)$, $f(x) = x$ in the proof.

Exercise. Check f fixes a and b in the proof.

4 2018.06.27. : Mladen Bestvina

Definition 4.1. If $(\Gamma, l, h), (\Gamma', l', h') \in \mathcal{CV}_n$, a *difference of markings* $\phi : \Gamma \rightarrow \Gamma'$ is a homotopy equivalence such that the diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{h} & \Gamma \\ & \searrow h' & \downarrow \phi \\ & & \Gamma' \end{array}$$

Remark. A conjugacy class in $\mathbb{F}_n$ corresponds to an inverse of loop in R . If $h : R \rightarrow \Gamma$ is a marking, and α is a conjugacy class (i.e. loop in R) then $h(\alpha)$ is a loop in Γ .

Definition 4.2. $l_\Gamma(\alpha)$ is the total length of edges the inversed loop $[h(\alpha)]$ crosses.

Example. $h(ab) = xy$, so $l_\Gamma(ab) = \frac{3}{2}$, $l_\Gamma(a^{-1}b) = \frac{1}{2}$.

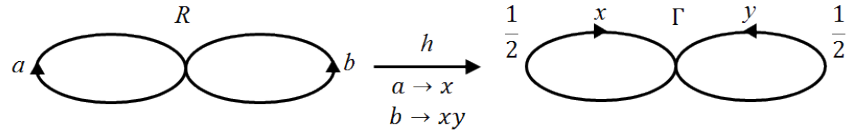


Figure 18: Example - total length.

Definition 4.3. Let $(\Gamma, l, h), (\Gamma', l', h') \in \mathcal{CV}_n$. The *Lipschitz metric* on $\mathcal{CV}_n$ is defined by

$$d(\Gamma, \Gamma') = \inf_{\phi} \log \sigma(\phi),$$

(ϕ is actually homotopy class) as $\phi : \Gamma \rightarrow \Gamma'$ ranges over difference of markings maps, and $\sigma(\phi)$ is the *Lipschitz constant* of ϕ .

Remark. 1. We can restrict consideration to maps that are linear on edges.

2. By applying Arzela–Ascoli theorem, the infimum is achieved.

Lemma 4.4. If $\phi : A \rightarrow B$ is K -Lipshitz, then for any conjugacy class α , $l_B(\alpha) \leq K \cdot l_A(\alpha)$.

Corollary 4.5. $d(A, B) \geq \log \frac{l_B(\alpha)}{l_A(\alpha)}$ for any conjugacy class α .

Example. Let A, B be given in the following figure. For $\phi : A \rightarrow B$, $\sigma(\phi) =$

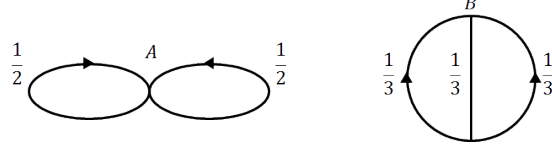


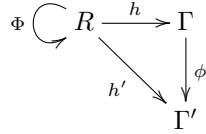
Figure 19: Example - Lipschitz metric.

$\frac{2/3}{1/2} = \frac{4}{3}$, and $d(A, B) \leq \log \frac{4}{3}$. By Corollary, $d(A, B) = \log \frac{4}{3}$. For $\psi : B \rightarrow A$, $\sigma(\psi) = \frac{1/2}{1/2} = \frac{3}{2}$, $d(B, A) = \log \frac{3}{2}$, as $l_B(ab^{-1}) = \frac{2}{3}$ and $l_A(ab^{-1}) = 1$.

Proposition 4.6. (1) $d(\Gamma_1, \Gamma_3) \leq d(\Gamma_1, \Gamma_2) + d(\Gamma_2, \Gamma_3)$,
(2) $d(\Gamma, \Gamma') \geq 0$ with equality iff $\Gamma = \Gamma'$,
(3) $d(\Gamma\Phi, \Gamma'\Phi) = d(\Gamma, \Gamma')$, for all $\Phi \in \text{Out}(\mathbb{F}_n)$.

Proof. (1) Use the fact that $\sigma(gf) \leq \sigma(g)\sigma(f)$.

(2) Observe that a difference of markings for Γ, Γ' and $\Gamma\Phi, \Gamma'\Phi$ are the same:



(3) Suppose $d(\Gamma, \Gamma') < 0$. Then there exists a difference of markings $\phi : \Gamma \rightarrow \Gamma'$ with $\sigma(\phi) < 1$. Then ϕ is not surjective, for volume reduces. Contradiction as ϕ is a homotopy equivalence.

Suppose $d(\Gamma, \Gamma') = 0$. Then have $\phi : \Gamma \rightarrow \Gamma'$, $\sigma(\phi) = 1$. ϕ must have slope 1 on every edge. Further, the images of edges can overlap only in points. It follows that ϕ identifies collection of points. If ϕ is not an isometry, this is impossible since Γ, Γ' have the same rank. So ϕ is an isometry, $\Gamma = \Gamma'$. ■

Definition 4.7. $\Gamma, \Gamma' \in \mathcal{CV}_n$. A *witness* is a conjugacy class α such that

$$d(\Gamma, \Gamma') = \log \frac{l_{\Gamma'}(\alpha)}{l_{\Gamma}(\alpha)}.$$

Note that we always have $\geq$.

Lemma 4.8. *Witnesses always exist.*

Definition 4.9. $\phi : \Gamma \rightarrow \Gamma'$ is an *optimal map* if $\log \sigma(\phi) = d(\Gamma, \Gamma')$.

Definition 4.10. The *tension graph* Δ_ϕ of an optimal map ϕ is the union of all edges where the slope is $\sigma(\phi)$. If v is a vertex of Δ_ϕ , denote by $T_v(\Delta_\phi)$ the set of *directions* at v .

Remark. Note that ϕ acts as a derivative $\phi_* : T_v(\Delta_\phi) \rightarrow T_{\phi(v)}(\Gamma')$.

Definition 4.11. A *train track structure* on a graph Δ is an equivalence relation on the set of directions $T_v(\Delta)$ at every vertex $v \in \Delta$.

Definition 4.12. The tension graph has a natural train track structure:

$$d \sim d' \iff \phi_*(d) = \phi_*(d').$$

The equivalence classes are *gates*.

For convention, draw equivalent directions so they are tangent to each other.

Example. Let ϕ, ψ be above.

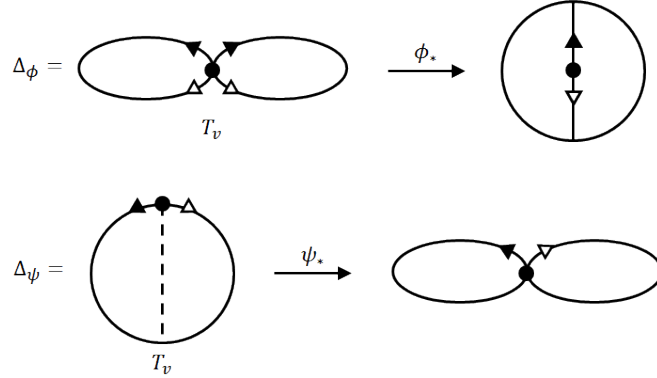


Figure 20: Example - tension graph and its natural train track structure. The equivalent gates are colored the same.

Definition 4.13. An immersed path or a loop in Δ_ϕ is *legal* if at every vertex the entering gate is distinct from the exiting gate.

Remark. 1. Δ_ϕ should attain a legal loop. Hence there is a vertex with ≥ 2 gates.

2. Observe that if α is a legal loop in Δ_ϕ then $l_{\Gamma'}(\alpha) = \sigma(\phi)l_\Gamma(\alpha)$.

3. If every vertex in Δ_ϕ has ≥ 2 gates, then Δ_ϕ supports a legal loop.

Proof of Lemma. We claim that if Δ_ϕ has a vertex with only one gate, then ϕ can be perturbed to an optimal map $\phi' : \Gamma \rightarrow \Gamma'$ such that $\Delta_{\phi'} \subsetneq \Delta_\phi$. The strategy is to keep approximating ϕ by optimal maps until no vertex in the tension graph has only one gate. Then use the above observations.

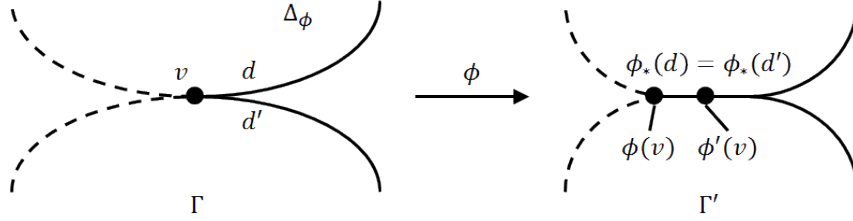


Figure 21: “Deleting” a vertex with only one gate from Δ_ϕ

Define $\phi'(v)$ to be near $\phi(v)$ in the direction $\phi_*(d)$, $d \in T_v(\Delta_\phi)$, and $\phi'(w) = \phi(w)$ for all other vertices. Extend to be linear on edges. Then the slope of ϕ' on edges of Δ_ϕ incident to v strictly decreases. The slope might increase on edges of $\Gamma - \Delta_\phi$ incident to v , but if the perturbation is small enough the new slope is still $< \sigma(\phi)$. So $\Delta_{\phi'} \subsetneq \Delta_\phi$. ■

Exercise. Complete the above proof; suppose Δ is a graph with a train track structure with ≥ 2 gate at every vertex. Then there is a legal loop in Δ that looks like one of the following:

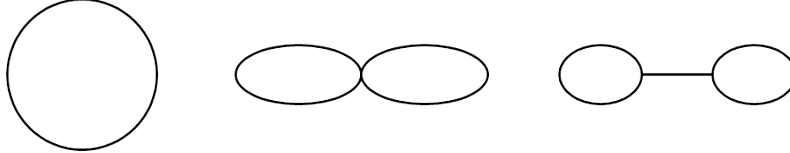


Figure 22: Possible forms of candidates. Train track structure is suggested by the picture.

Remark. There is an algorithm to compute $d(\Gamma, \Gamma')$ without first finding an optimal maps: Enumerate all loops in Γ that have one of the 3 forms above. For each such loop α , compute $\log \frac{l_{\Gamma'}(\alpha)}{l_\Gamma(\alpha)}$. Then the maximal value is precisely $d(\Gamma, \Gamma')$.

Definition 4.14. Let A, B be distinct vertices in a graph G . The *maximum flow* is the largest number of paths from A to B that are disjoint except for A and B . The *minimum cut* is the smallest number of vertices that separate A from B .

Theorem 4.15 (Menger). *The maximum flow is equal to the minimum cut.*

Definition 4.16. If $\phi : X \rightarrow X$ is an isometry of a metric space X , define the *translation length* of ϕ by

$$T(\phi) = \lim_{n \rightarrow \infty} \frac{d(x_0, \phi^n(x_0))}{n}.$$

Remark. The limit always exists independent in x_0 . Indeed, if y_0 is another point, then

$$d(x_0, \phi^n(x_0)) \leq d(x_0, y_0) + d(y_0, \phi^n(y_0)) + d(\phi^n(y_0), \phi^n(x_0)),$$

so

$$d(x_0, \phi^n(x_0)) - d(y_0, \phi^n(y_0)) \leq d(x_0, y_0) + d(y_0, x_0),$$

and similarly

$$d(y_0, \phi^n(y_0)) - d(x_0, \phi^n(x_0)) \leq d(x_0, y_0) + d(y_0, x_0).$$

Hence the quantity $|d(x_0, \phi^n(x_0)) - d(y_0, \phi^n(y_0))|$ is bounded, the limit is independent in x_0 . The limit exists by the below Polya lemma.

Lemma 4.17 (Polya). *If $a_n \geq 0$ and $a_{m+n} \leq a_m + a_n$, then $\lim_n \frac{a_n}{n}$ exists.*

Definition 4.18. The *displacement* of ϕ is defined by

$$D(\phi) = \inf_{x \in X} d(x, \phi(x)).$$

Observe that $D(\phi) \geq T(\phi)$.

Definition 4.19. 1. ϕ is *elliptic* if $D(\phi) = 0$ and ϕ is *realized*, i.e. ϕ fixes a point.

2. ϕ is *hyperbolic* if $D(\phi) > 0$ and is realized.

3. ϕ is *parabolic* if $D(\phi)$ is not realized.

Proposition 4.20 (Nielsen realization). *$\Phi \in \text{Out}(\mathbb{F}_n)$ is elliptic in $\mathcal{CV}_n$ if and only if Φ has finite order.*

Example. $a \mapsto a, b \mapsto ab$ is parabolic. Clearly ϕ has infinite order. To show $D(\phi) = 0$, set Γ_ϵ be a rose with length ϵ and $1 - \epsilon$. Then $d(\Gamma_\epsilon, \Gamma_\epsilon \Phi) \rightarrow 0$ as $\epsilon \rightarrow 0$.

Theorem 4.21. *There are parabolic automorphisms with $D(\phi) > 0$.*

5 2018.06.27. : Kenichi Ohshika

Review on Teichmüller space/theory

Let S be a closed orientable surface of genus ≥ 2 . Then S admits a hyperbolic metric; set

$$\mathcal{H}(S) = \{\text{hyperbolic metrics on } S\}.$$

$g_1, g_2 \in \mathcal{H}(S)$ are said to be *equivariant* if there is a diffeomorphism $f : S \rightarrow S$ homotopic to the identity such that $g_2 = f^*g_1$. Let

$$\mathcal{T}(S) = \mathcal{H}(S) / \sim.$$

For $(S, m) \in \mathcal{H}(S)$, the universal cover $(\tilde{S}, \tilde{m}) \simeq \mathbb{H}^2$, so $(S, m) \simeq \mathbb{H}^2 / \varphi(\pi_1(S))$ where $\varphi : \pi_1(S) \rightarrow \text{Isom}^+(\mathbb{H}^2) = \text{PSL}_2(\mathbb{R})$ is faithful discrete representation. Then the equivariant metrics correspond to the conjugacy on the representation in $\text{PSL}_2(\mathbb{R})$, so $\mathcal{T}(S)$, considered as the set of conjugacy classes of faithful discrete representations of $\pi_1(S)$ into $\text{PSL}_2(\mathbb{R})$, has the natural topology conjugate from the representation space.

Definition 5.1. Let Σ_1, Σ_2 be Riemann surfaces, $h : \Sigma_1 \rightarrow \Sigma_2$ be a homeomorphism. We define the *Beltrami coefficient* at $z \in \Sigma_1$ by

$$M_h(z) = \frac{h_{\bar{z}}(z)}{h_z(z)},$$

where

$$h_{\bar{z}} = \frac{1}{2} \left(\frac{\partial h}{\partial x} - i \frac{\partial h}{\partial y} \right), \quad h_z = \frac{1}{2} \left(\frac{\partial h}{\partial x} + i \frac{\partial h}{\partial y} \right),$$

in the distribution sense. Define the *dilatation* at z by

$$K_h(z) = \frac{1 + |M_h(z)|}{1 - |M_h(z)|},$$

and the *maximum dilatation* by

$$K_h = \sup_{z \in \Sigma_1} K_h(z).$$

Exercise. Show that $|M_h(z)| < 1$.

Exercise. For $h_1 : \Sigma_1 \rightarrow \Sigma_2$ and $h_2 : \Sigma_2 \rightarrow \Sigma_1$, $K_{h_2 \circ h_1}(z) \leq K_{h_2}(h_1(z)) \cdot K_{h_1}(z)$.

Definition 5.2. h is *quasi-conformal* if $K_h < \infty$.

Theorem 5.3 (Teichmüller). *Let Σ_1, Σ_2 be Riemann surfaces, $h : \Sigma_1 \rightarrow \Sigma_2$ be a homeomorphism. Then there is a quasi-conformal $h \simeq f : \Sigma_1 \rightarrow \Sigma_2$ such that for any homeomorphism $h \simeq g : \Sigma_1 \rightarrow \Sigma_2$, $K_f \leq K_g$ with equality iff $g = f$. Such f is called the Teichmüller map.*

Definition 5.4 (Teichmüller). Now give an equivalence relation on the set of (Σ, f) , where Σ is a Riemann surface and $f : S \rightarrow \Sigma$ is an orientation preserving homeomorphism, by $(\Sigma_1, f_1) \sim (\Sigma_2, f_2) \iff f_1 \cdot f_2^{-1} : \Sigma_2 \rightarrow \Sigma_1$ is homotopic to isometry. Let $\mathcal{T}(S)$ be the set of equivalence classes of such (Σ, f) . Define a metric $d_{\mathcal{T}}$ on $\mathcal{T}(S)$ by

$$\begin{aligned} d_{\mathcal{T}}(x, y) &= \frac{1}{2} \log \inf \{K_g \mid g \simeq f_1 \cdot f_2^{-1}\} \\ &= \frac{1}{2} \log K_g, \end{aligned}$$

for $x, y \in \mathcal{T}(S)$, where $g \simeq f_1 \cdot f_2^{-1}$ is the Teichmüller map above.

Definition 5.5. For $[(\Sigma, f)] \in \mathcal{T}(S)$, $\text{QD}(\Sigma)$ is the space of (holomorphic) quadratic differentials. Define $t : \text{QD}(\Sigma) \rightarrow \mathcal{T}(S)$ by the solution of

$$\frac{h_{\bar{z}}}{h_z} = \frac{\bar{\varphi}}{|\varphi|},$$

for $\varphi \in \text{QD}(\Sigma)$. Note that $\text{QD}(\Sigma) \simeq \mathbb{R}^{6g-6}$ where g is the genus of S .

Definition of “deformation spaces” of Kleinian groups

Definition 5.6. Let $I = (0, 1)$. Let

$$\mathcal{H}^3(S) = \{\text{complete hyperbolic metrics on } S \times I\}.$$

Define $m_1 \sim m_2 \iff$ there exists $h : S \times I \rightarrow S \times I$ isotopic to the identity such that $m_2 = h^*m_1$. Then $\mathcal{H}(S) = \mathcal{H}^3(S)/\sim$ gives a generalization.

Definition 5.7. Define $\mathcal{AH}(S)$ by the set of the faithful discrete representations of $\pi_1(S)$ into $\text{Isom}^+(\mathbb{H}^3) = \text{PSL}_2(\mathbb{C})$ mod out the isotopy.

Theorem 5.8 (Baer–Nielsen). *If there is a homotopy equivalence between two closed surface, then it must be homotopic to a homeomorphism.*

Definition 5.9. The *Kleinian group* is a discrete subgroup of $\text{PSL}_2(\mathbb{C})$.

Definition 5.10. For $\varphi : \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{C})$, a faithful discrete representation $\psi : \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{C})$ is said to be *quasi-conformal deformation* of φ if there is an orientation preserving homeomorphism $h : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ such that

$$\left\| \frac{h_{\bar{z}}}{h_z} \right\|_{\infty} < 1,$$

where $\|\cdot\|_{\infty}$ denotes the essential supremum, and for $z \in \widehat{\mathbb{C}}$ and $g \in \pi_1(S)$,

$$\psi(g)z = h \circ \varphi(g) \circ h^{-1}z.$$

(Baer simultaneous uniformization) For $x \in [(\Sigma, f)] \in \mathcal{T}(S)$, note that a Fuchsian representation $\varphi : \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{R}) \subset \text{PSL}_2(\mathbb{C})$ is given. The *limit set* of $\varphi(\pi_1(S))$ is defined by

$$\Lambda_{\varphi(\pi_1(S))} = \overline{\{z \in \widehat{\mathbb{C}} \mid \exists \gamma \neq \text{id s.t. } \varphi(\gamma)z = z\}} = \widehat{\mathbb{R}}.$$

Note that $\varphi(\pi_1(S))$ acts properly discontinuously on $D_{\pm}$, where $D_{\pm}$ are defined as in the below figure. Then

$$D_{\pm}/\varphi(\pi_1(S)) = \Sigma_{\pm}.$$

Consider two Teichmüller maps $h_{\pm} : \Sigma \rightarrow \Sigma_{\pm}$. Let $\mu_{\pm}$ be their Beltrami coefficient. Let $\mu_{\pm} \leftarrow \widetilde{\mu}_{\pm}$.

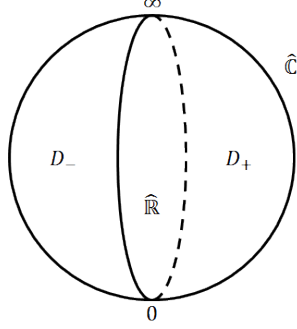


Figure 23: Construction of quasi-conformal deformation.

Let $h : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be quasi-conformal homeomorphism with

$$\frac{h_{\bar{z}}}{h_z} = \begin{cases} \widetilde{\mu}_+, & \text{on } D_+, \\ \widetilde{\mu}_-, & \text{on } D_-, \end{cases}$$

i.e. h combine h_+, h_- “geometrically”. Then $\psi = h\varphi h^{-1}$ is a quasi-conformal deformation.

Conversely, any quasi-conformal deformation of φ is obtained in this way:

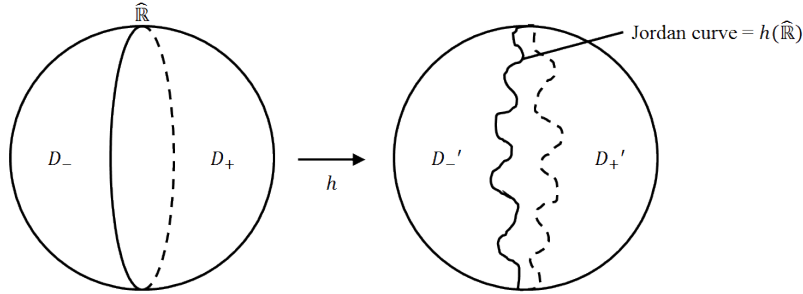


Figure 24: Quasi-conformal deformation defines quasi-Fuchsian.

$h_{\pm} : D_{\pm}/\varphi(\pi_1(S)) \rightarrow D'_{\pm}/\varphi(\pi_1(S))$. We define

$$\mathcal{QF}(S) = \mathcal{T}(S) \times \mathcal{T}(\overline{S}) \xrightarrow[\simeq]{} \{\text{quasi-conformal deformation of } \varphi\}/\text{conj.}$$

Note that $\mathcal{QF}(S) \simeq \mathbb{R}^{12g-12}$ and $\mathcal{T}(S)$, $\mathcal{T}(\overline{S})$ correspond to D_+, D_- , respectively. The element of $\mathcal{QF}(S)$ is called *quasi-Fuchsian*.

Exercise. Show that $\Lambda_{\varphi(\pi_1)(S)} = \widehat{\mathbb{R}}$.

Remark. 1. $\mathcal{H}(S) \hookrightarrow \mathcal{AH}(S)$, mapping a hyperbolic metric $(S \times I, m)$ to a universal cover $(\tilde{S} \times I, \tilde{m}) \simeq \mathbb{H}^3 \curvearrowright \pi_1(S)$ by isometries, so we have a faithful discrete representation $\varphi : \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{C})$.

2. The above map is surjective. Indeed, for given $\psi : \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$, so $\pi_1(\mathbb{H}^3/\psi(\pi_1(S))) \simeq \pi_1(S)$. So by Whitehead's theorem, $S \simeq_{h.e.} \mathbb{H}^3/\psi(\pi_1(S))$ by sending loops to loops and extending.

3. $\mathcal{QF}(S) \hookrightarrow \mathcal{AH}(S)$ naturally.

Exercise. Show that the map defined as in the above Remark 1 is injective.

Theorem 5.11 (Bonahon). *For any $\psi \in \mathcal{AH}(S)$, $\mathbb{H}^3/\psi(\pi_1(S)) \simeq S \times (0, 1)$. (by using the technique of Thurston.)*

Corollary 5.12. $\mathcal{H}(S) = \mathcal{AH}(S)$.

Theorem 5.13 (Ben, Maslcit?). $\mathcal{AH}(S) \setminus \mathcal{QF}(S) \neq \emptyset$. *Actually, it contains uncountably many points.*

Theorem 5.14 (Brack Browder?). $\mathcal{AH}(S) = \overline{\mathcal{QF}(S)}$, *here the closure is taken in the space of representations mod out conjugates.*

Theorem 5.15 (Sullivan). $\mathrm{Int}(\mathcal{AH}(S)) = \mathcal{QF}(S)$, *here the interior is taken in the space of representations mod out conjugates.*

Theorem 5.16 (McMullen). $\mathcal{AH}(S)$ *is not a manifold.*

More general cases

Definition 5.17. Let M be a compact orientable 3-manifold.

1. M is *irreducible* if any embedded sphere in M bounds a 3-ball.
2. M is *atoroidal* if any embedded incompressible torus is homotopic into ∂M .
3. A surface $S \subset M$ is said to be *incompressible* if $\pi_1(S)$ injects to $\pi_1(M)$ by a homomorphism induced by the inclusion.

(Continued)

6 2018.06.28. : Koji Fujiwara

Definition 6.1. We use the *upper hyperplane model* $\mathbb{H}^2 = \{(x, y) \mid y > 0\} \subset \mathbb{C}$, with a given metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$. Also, any element in $\mathrm{SL}(2, \mathbb{R})$ isometrically acts on $\mathbb{H}^2$ by the linear fractional transformation: $z \mapsto \frac{az+b}{cz+d}$.

Example. Geodesics in $\mathbb{H}^2$:

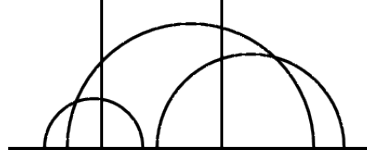


Figure 25: Example - geodesics in $\mathbb{H}^2$.

Exercise. Prove that elements in $\mathrm{SL}(2, \mathbb{R})$ defines an isometry. [Hint: $\mathrm{SL}(2, \mathbb{R}) = \langle \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & 1/b \end{pmatrix} \mid a, b \in \mathbb{R} \rangle$.]

Remark. $\mathrm{Isom}^+(\mathbb{H}^2) \simeq \mathrm{PSL}(2, \mathbb{R})$.

Definition 6.2. A discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ is called *Fuchsian*.

Remark. Let $\Gamma < \mathrm{PSL}(2, \mathbb{R})$ be a discrete, torsion-free subgroup. As $\mathbb{H}^2/\Gamma$ is 2-manifold, either Γ is a surface group (i.e. $\mathbb{H}^2/\Gamma$ is compact) or Γ is a free group (i.e. $\mathbb{H}^2/\Gamma$ is not compact).

Exercise. Prove that $\mathbb{H}^2 \rightarrow \mathbb{H}^2/\Gamma$ is a covering map.

Theorem 6.3 (Ihara, 1966). *Discrete, torsion-free $\Gamma < \mathrm{SL}(2, \mathbb{Q}_p)$ is free.*

Serre. Find a group action $\mathrm{SL}(2, \mathbb{Q}_p) \curvearrowright T$, where T is a Bruhat–Tits tree. Then $T \rightarrow T/\Gamma$ is a covering, so that T/Γ is a graph. Thus $\pi_1(T/\Gamma) \simeq \Gamma$ is free, as Γ acts freely on a tree. ■

Definition 6.4. A *tree* is a simplicial graph which is connected and simply connected (i.e. no embedded S^1).

Develop the Bass–Serre theory. Note that $\mathrm{SL}(2, \mathbb{Z}) < \mathrm{SL}(2, \mathbb{R})$ acts on the below tree:

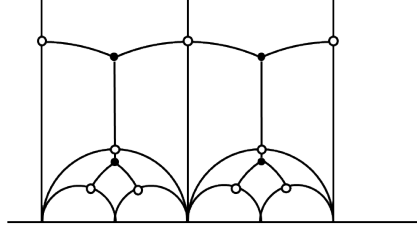


Figure 26: $\mathrm{SL}(2, \mathbb{Z})$ acts on a tree.

This action encodes an algebraic information

$$\mathrm{SL}(2, \mathbb{Z}) \simeq \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6,$$

where an edge is fixed by $\mathbb{Z}/2$, a white vertex is fixed by $\mathbb{Z}/4$, a black vertex is fixed by $\mathbb{Z}/6$, and the quotient is one edge. On the other hand, if $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$ acts on a simplicial tree by isometries, then there is a fixed point.

Definition 6.5. The property described above is called *property FA*.

Example. 1. Let Σ_g be a closed orientable surface of genus g , and let $\mathrm{MCG}(\Sigma_g)$ be the mapping class group of Σ_g . If $g \geq 2$, $\mathrm{MCG}(\Sigma_g)$ has property FA (Culler–Vogtmann), but $\mathrm{MCG}(\Sigma_g) \simeq \mathrm{SL}(2, \mathbb{Z})$ does not.

2. $\mathrm{Out}(\mathbb{F}_n)$ has property FA for $n \geq 3$. (Bogopoloski)

Note. Note that $\mathrm{MCG}(\Sigma) = \mathrm{Homeo}^+(\Sigma)/\text{isotopy} \simeq \mathrm{Out}(\pi_1(\Sigma))$ acts on the Teichmüller space of Σ (which is diffeomorphic to $\mathbb{R}^{6g-6}$) by isometries.

1. *Dehn twist* along a simple closed curve α (i.e. cut the surface along α then twist it by 360 degrees, then glue it back) generates MCG .

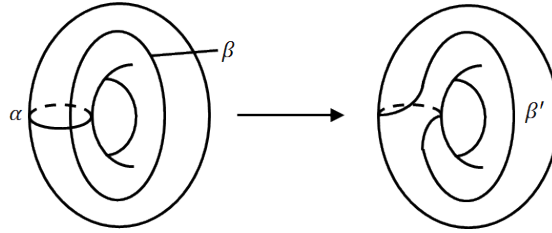


Figure 27: Dehn twist along a simple closed curve α .

2. Pseudo–Anson element (typical) does not have any invariant disjoint simple closed curves.

Definition 6.6. A geodesic metric space X is a *quasi-tree* if it is *quasi-isometric* to a simplicial tree T , i.e. there exist $L > 0$, $f : X \rightarrow T$, $g : T \rightarrow X$ such that

$$\frac{|x - y|}{L} - L \leq |f(x) - f(y)| \leq L|x - y| + L, \forall x, y \in X,$$

$$\frac{|t - s|}{L} - L \leq |g(t) - g(s)| \leq L|t - s| + L, \forall t, s \in T.$$

Example. 1. $\mathbb{Z}^2$ with l^1 -metric is quasi-isometric to $\mathbb{R}^2$, which is not a quasi-tree.

2. The Farey graph F is quasi-isometric to a tree, hence is a quasi-tree. If we remove a (closed) edge from F then F gets disconnected. Then F is a quasi-tree.

3. “bottle neck property” (Monning) characterizes a quasi-tree. For example, in a tree, removing a point disconnects the tree.

Exercise. Prove that Farey graph is not quasi-isometric to its dual binary tree.

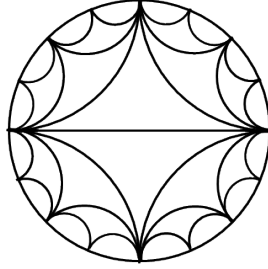


Figure 28: Farey graph.

Remark. 1. For fixed $x \in X$, the map $\gamma \in \text{MCG}(\Sigma_g) \mapsto \gamma(x) \in X$ is called an *orbit map*.

2. γ is hyperbolic if there exists $C > 0$ such that $\forall n > 0$, $|\gamma^n(x) - x| \geq Cn$. (e.g. $\gamma^n(x)$ are on a geodesic.)

Theorem 6.7 (Bestvina–Bromberg–F, 2018). $\text{MCG}(\Sigma_g)$, $g \geq 1$, *isometrically acts on a finite product of quasi-trees*, $X = X_1 \times \cdots \times X_n$, *preserving the decompositions, which gives an equivalent quasi-isometric embedding of $\text{MCG}(\Sigma_g)$ in X by an orbit map, with respect to a word metric of $\text{MCG}(\Sigma_g)$. Also, any element $\gamma \in \text{MCG}(\Sigma_g)$ of infinite order is hyperbolic on X .*

Remark. No estimate of n in terms of g .

Theorem 6.8 (Farb–Lubotzky–Mozes). *Any $\gamma \in \text{MCG}(\Sigma)$ of infinite order is not distorted in $\text{MCG}(\Sigma)$, i.e. the word norm $|\gamma^n|$ grows linearly on $n > 0$.*

Theorem 6.9 (Mangahas). *Each finitely generated subgroup $\Gamma < \text{MCG}(\Sigma)$ is either virtually abelian or has uniform exponential growth, i.e. there exists $C(\Sigma) > 0$ such that for every finite $S \subset \text{MCG}(\Sigma)$ with $\langle S \rangle = \Gamma$ and every $n > 0$, $\#(S^n) \geq e^{Cn}$.*

Theorem 6.10 (Kida, Hamenstadt). *The asymptotic dimension of $\text{MCG}(\Sigma) \leq n$. In particular, it is finite. (This implies Novikov conjecture on Σ .)*

Remark. 1. Any tree has asymptotic dimension 1, and quasi-isometry preserves the asymptotic dimension, so any quasi-tree has asymptotic dimension 1.

2. Many groups have unbounded actions on quasi-tree, i.e. an orbit map is unbounded. (e.g. $\text{Out}(\mathbb{F}_n)$ for $n \geq 2$.)

Definition 6.11. A group G is *hyperbolic* if it acts properly and cocompactly on a δ -hyperbolic space X by isometries (in the sense of Gromov).

Example. Free groups, $\text{SL}(2, \mathbb{Z})$, and surface groups of genus ≥ 2 .

Theorem 6.12 (BBF). *Let G be a residually finite, hyperbolic group. Then G acts on a finite product of quasi-trees by isometries such that the orbit maps are quasi-isometric embedding. Say it is property (T).*

Example. Let G be a uniform lattice in a Lie group $\text{Sp}(2n, 1)$ of rank 1, $n \geq 1$. Then G is residually finite, hyperbolic group. It has property (T), therefore, if it acts on a finite product of simplicial trees, or more generally, a $\text{CAT}(0)$ cube complex, then it must fix a point (Niblo–Reeves).

Remark. Let Γ be discrete, torsion-free Fuchsian group. Now we construct a group action of Γ on a quasi-tree. Let $\Gamma \curvearrowright \mathbb{H}^2$ by isometries. Assume there exists $g \in \Gamma$ which is hyperbolic on $\mathbb{H}^2$ with an invariant geodesic $\gamma \subset \mathbb{H}^2$, which is an *axis*. Let $G = \{a\gamma \mid a \in \Gamma\}$ be the set of disjoint lines, and let $\Gamma \curvearrowright G$. For each $A \in G$, define $\pi_A : \mathbb{H}^2 \rightarrow A$ be the nearest point projection. Then there exists $L > 0$ such that for any $B \in G$ distinct from A , $\text{diam}\pi_A(B) \leq L$. Define

$$d_C(A, B) = \text{diam}\pi_C(A \cup B).$$

Fix $K > 0$, We say $C \in G$ is *between* $A, B \in G$ if $d_C(A, B) \leq K$. Join A, B by edges at $\pi_A(B)$ and $\pi_B(A)$, if there is no C between A, B .

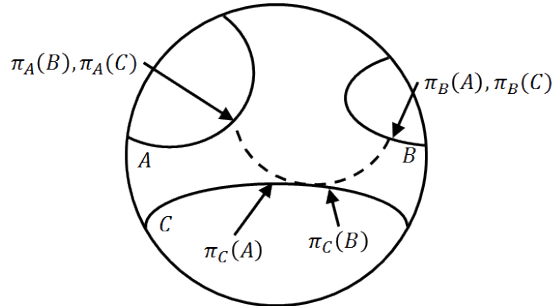


Figure 29: Construction of quasi-tree.

Theorem 6.13. *The resulting graph is quasi-tree for large K .*

In the above figure, $\pi_A(B)$ etc looks like a point. The dashed line is the shortest segment between A, B in $\mathbb{H}^2$. C and this segment stay close for a long time, therefore $d_C^\pi(A, B)$ is large, while $d_A^\pi(B, C)$ and $d_B^\pi(A, C)$ are small.

Definition 6.14. For $\delta \geq 0$, δ -hyperbolic space X is a geodesic space such that for all geodesic triangle $\triangle(\alpha, \beta, \gamma)$ is δ -thin, i.e. $\alpha \subset N_\delta(\beta \cup \gamma)$, $\beta \subset N_\delta(\alpha \cup \gamma)$, $\gamma \subset N_\delta(\alpha \cup \beta)$.

Definition 6.15. G acts properly on X if for any $x \in X$ and $R > 0$, $\{g \in G \mid |x - g(x)| < R\}$ is finite.

Example. 1. $\mathbb{H}^2$ is 2-hyperbolic space. Gauss–Bonnet theorem says $\triangle(\alpha, \beta, \gamma)$ has area $< \pi$, so $\triangle(\alpha, \beta, \gamma)$ is 2-thin.

2. Any tree is 0-hyperbolic space.
3. $\pi_1(\Sigma_g)$, $g \geq 2$ is a hyperbolic group, as $\pi_1(\Sigma_g) \curvearrowright \mathbb{H}^2$.
4. $\mathbb{F}_2$ is a hyperbolic group, as $\mathbb{F}_2$ acts on a 4-regular tree.
5. Any bounded set is δ -hyperbolic space.
6. $\mathbb{Z}$ is hyperbolic, but $\mathbb{Z}^2$ is not hyperbolic.

Definition 6.16. Let X be δ -hyperbolic space. Then for a geodesic $\gamma \subset X$, $\pi_\gamma : X \rightarrow \gamma$ is the nearest points projection.

Lemma 6.17. *If X is δ -hyperbolic, then for all $x \in X$, $\text{diam} \pi_\gamma(x) \leq 100\delta$.*

Lemma 6.18. *Let X be δ -hyperbolic space, and $\gamma, \beta \subset X$ be geodesics. If $\text{diam} \pi_\alpha(\beta)$ is large, say $\geq 1000\delta$, then $\pi_\alpha(\beta) \subset N_{20\delta}(\beta)$.*

Proof. For $p, q \in \beta$, let $\bar{p} \in \pi_\alpha(p)$, $\bar{q} \in \pi_\alpha(q)$. As $\triangle(p, \bar{p}, \bar{q})$ is δ -thin, $\bar{p}\bar{q} \subset N_\delta(p\bar{p} \cup p\bar{q})$. So $\bar{p}\bar{q} \subset N_{10\delta}(p\bar{q})$. Also $\triangle(p, q, \bar{q})$ is δ -thin, so $p\bar{q} \subset N_\delta(pq \cup q\bar{q})$. Now $|\bar{p} - \bar{q}| \geq 1000\delta$. So $p\bar{q} \subset N_{10\delta}(pq)$, $\bar{p}\bar{q} \subset N_{20\delta}(pq)$. ■

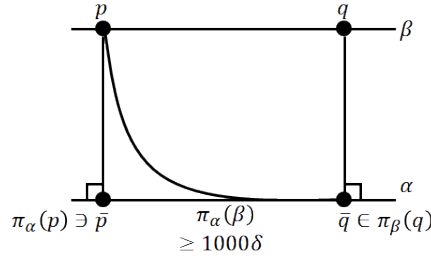


Figure 30: δ -thin triangle.

Suppose G acts properly on δ -hyperbolic space X by isometries. Fix $g \in G$ hyperbolic on X , i.e. there exists a geodesic $\gamma \subset X$ invariant by g . Then g acts on γ as a translation with translation length $\tau(g) > 0$.

Definition 6.19. A *quasi-geodesic* in X is a quasi-isometric embedding of $\mathbb{R}$ into X .

Definition 6.20. α, β are *parallel*, denoted $\alpha \sim \beta$, if there exists $L > 0$ such that $\beta \subset N_L(\alpha)$ (or equivalently, $\alpha \subset N_L(\beta)$).

Proposition 6.21. There is $\xi > 0$ such that for all $a \in G$, either $\text{diam} \pi_\gamma(a\gamma) \leq \xi$ or γ is parallel to $a\gamma$.

Lemma 6.22. Let X be a δ -hyperbolic space, and α, β be ∞ -geodesics. Then $\alpha \sim \beta$ iff $\alpha \subset N_{20\delta}(\beta)$, $\beta \subset N_{20\delta}(\alpha)$.

Proof. ($\Leftarrow$) Trivial.

($\Rightarrow$) As $\beta \subset N_L(\alpha)$, $\text{diam} \pi_\alpha(\beta) = \infty$. By applying lemma, $\alpha \subset N_{20\delta}(\beta)$. ■

Proof of Proposition. Suppose $\pi_\gamma(a\gamma)$ is large ($\geq \xi \gg \tau(g)$, decided later). Then $a\gamma$ is an axis for aga^{-1} , and $\tau(aga^{-1}) = \tau(g)$. For large $L > 0$ and all $0 < n \leq L$, $|(aga^{-1})^{-n}g^n(p) - p| \leq 100\delta$. By using properness, there exist $n \neq m \leq L$ such that

$$(aga^{-1})^{-n}g^n = (aga^{-1})^{-m}g^m, \quad ag^Na^{-1} = g^N,$$

where $N = m - n \neq 0$. Now γ is an axis for g^N , $a\gamma$ is an axis for $ag^Na^{-1} = g^N$, so $\gamma \sim a\gamma$. ■

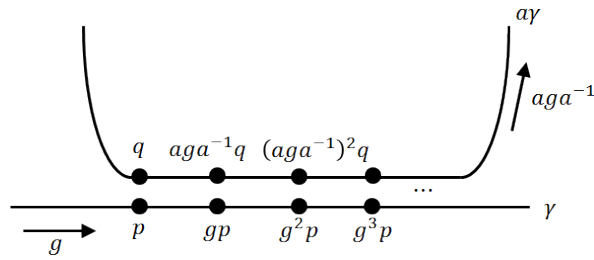


Figure 31: $\gamma, a\gamma$ are parallel.

Exercise. If α, β are axes for g , then $\alpha \sim \beta$.

7 2018.06.28. : Kenichi Ohshika

(continued) Let M be a compact orientable irreducible atoroidal 3-manifold.

Theorem 7.1 (Scott). *Let N be a non-compact 3-manifold. Assume that $\pi_1(N)$ is finitely generated. Then there exists a compact 3-manifold $C \hookrightarrow N$ such that the inclusion is homotopy equivalence.*

Caution. 1. The assumption for $\pi_1(N)$ above is essential.

2. The above theorem describes the special property of 3-manifold; if the fundamental group is finitely generated, then it is finitely presented. But this is not true in general; there is a counter-example for 4-manifold.

Corollary 7.2. *Suppose G is a finitely generated torsion-free Kleinian group. Then $\mathbb{H}^3/G$ contains a compact orientable irreducible atoroidal 3-manifold $C \hookrightarrow \mathbb{H}^3/G$. Such C is called the compact core or Scott core.*

Remark. If a hyperbolic 3-manifold contains an incompressible torus, then it must correspond to a “torus cusp”:

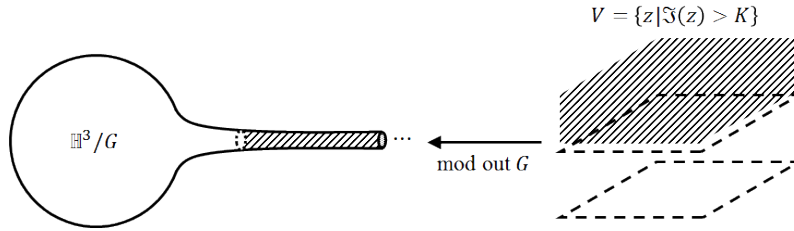


Figure 32: Torus cusp.

Any level set corresponds to torus and as metric decreases for large K , the torus “gets smaller”.

Definition 7.3. 1. $A \in \text{PSL}_2(\mathbb{C})$ is *elliptic* if $A \sim \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$, $|\omega| = 1$.

2. A is *parabolic* if $A \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

3. A is *hyperbolic* if $A \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $|\lambda| > 1$.

If a Kleinian group $G \supset \mathbb{Z} \times \mathbb{Z}$, we can conjugate G with $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$.

Theorem 7.4 (Bonahon, Agol, Calegari–Gabai). *Let G be a finitely generated torsion-free Kleinian group, and $C \hookrightarrow \mathbb{H}^3/G$ be a compact core. Then $\text{Int}C$ is homeomorphic to $\mathbb{H}^3/G$ by a homeomorphism homotopic to the inclusion.*

Theorem 7.5 (Thurston). *Let M be a compact orientable irreducible atoroidal 3-manifold with $\partial M \neq \emptyset$. Then $\text{Int}M$ admits a complete “geometrically finite” hyperbolic metric without a cusp of rank 1.*

Definition 7.6. $\mathcal{H}(M) = \{\text{hyperbolic metrics on } \text{Int } M\}/\text{isotopy}.$

$\mathcal{AH}(M) = \{\text{faithful discrete representation of } \pi_1(M) \text{ into } \text{PSL}_2(\mathbb{C})\}/\text{conj}.$

Remark. 1. $\mathcal{H}(M) \subset \mathcal{AH}(M)$ by holonomy, in general the equality does not hold.

2. $\mathcal{AH}(M)$ is the disjoint union of $\mathcal{H}(N, f) = \mathcal{H}(N)$ among the homotopy equivalences $f : M \rightarrow N$, and a compact orientable irreducible atoroidal 3-manifold N . Here $f_i : M \rightarrow N_i$ are equivalent if $f_1 \circ f_2^{-1}$ is homotopic to a homeomorphism.

3. Suppose ∂M is incompressible. If M contains a 3-dimensional surface manifold $\Sigma \times I$, then $\Sigma \times \{0, 1\} \subset \partial M$ is not contractible in ∂M . Let V be a solid torus of M . Then $\partial V \cap \partial M$ is a union of non-contractible amuls on ∂V . Cut V out and glue it back after changes of order of amul, then it gives a homotopy equivalence.

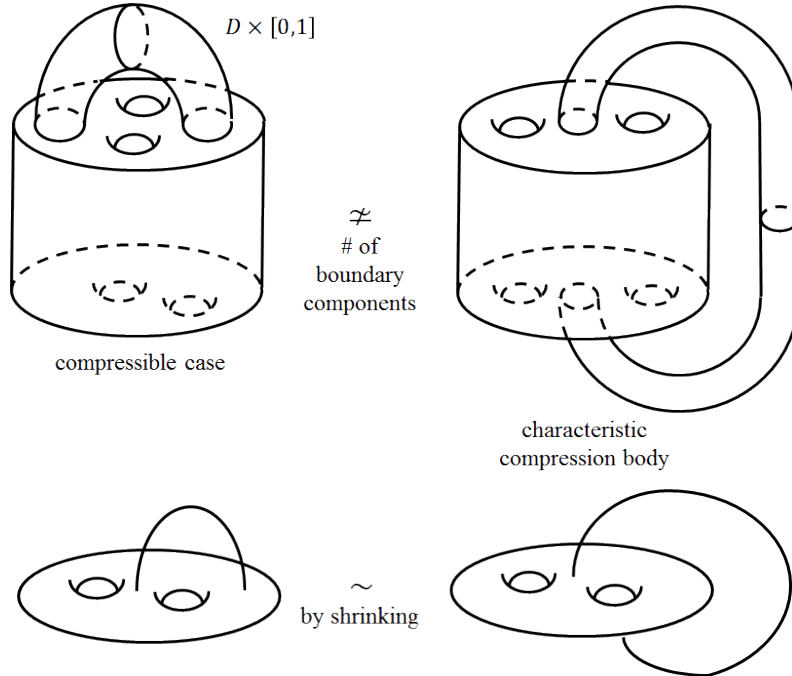


Figure 33: Example - characteristic compression body.

Theorem 7.7 (Jaco–Shalen, Juhannson). *Suppose ∂M is incompressible. Then any homotopy equivalence $f : M \rightarrow N$ is obtained by iterate flipping and shuffling.*

Theorem 7.8 (McCullough–Miller). *Flipping, shuffling and characteristic compression body generate all homotopy equivalence.*

Definition 7.9. Let G be finitely generated torsion-free Kleinian group. The *limit set* of G is defined by

$$\Lambda_G = \overline{\{z \in \widehat{\mathbb{C}} = \partial_\infty \mathbb{H}^3 \mid \exists \gamma \neq \text{id} \in G \text{ s.t. } \gamma z = z\}}.$$

Define the *Nielsen convex hull* H_G by the smallest convex closed set in $\mathbb{H}^3$ containing all geodesics whose end-points lie on Γ_G .

Note. 1. If $\gamma z = z$, then $g\gamma g^{-1}$ fixes gz . Hence Γ_G is G -invariant, and also H_G is G -invariant.

2. $H_G/G \subset \mathbb{H}^3/G$ is convex closed.

Definition 7.10. A finitely generated torsion-free Kleinian group G is *geometrically finite* if $\text{vol}(H_G/G)$ is finite.

Example. 1. Any Fuchsian group G is geometrically finite. Indeed, as H_G/G is totally geodesic and 2-dimensional, $\text{vol}(H_G/G) = 0$.

2. Any subgroup of $\mathcal{QF}(S)$ is geometrically finite.

3. Non-example is given by Bers (1970s).

Definition 7.11. A Kleinian group G is *parabolic* if any non-identity elements in G are parabolic.

Definition 7.12. A Kleinian group G is said to be *minimally parabolic* if any maximal parabolic subgroup is rank 2.

Definition 7.13. For faithful discrete representation $\varphi : \pi_1(M) \rightarrow \text{PSL}_2(\mathbb{C})$, the *quasi-conformal deformation* of φ is defined as follows: for a quasi-conformal map $h : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ with

$$\left\| \frac{h_{\bar{z}}}{h_z} \right\|_\infty < 1,$$

if for any $\gamma \in \pi_1(M)$ $h\varphi(\gamma)h^{-1} \in \text{PSL}_2(\mathbb{C})$, then $\psi = h\varphi(\gamma)h^{-1}$ is a quasi-conformal deformation of φ . Define

$$\mathcal{QH}(M) = \{\text{quasi-conformal deformation of } \varphi\} / \text{conj.}$$

Remark. $\mathcal{QH}(M) \subset \mathcal{H}(M) \subset \mathcal{AH}(M)$. $\mathcal{H}(M) \setminus \mathcal{QH}(M) \neq \emptyset$ in general.

Definition 7.14. $\Omega_\varphi = \widehat{\mathbb{C}} \setminus \Gamma_{\varphi(\pi_1(M))}$.

Remark. $\Omega_\varphi/\varphi(\pi_1(M))$ is actually a Riemann surface of finite type. By Manden, $\mathcal{T}(\Omega_\varphi/\varphi(\pi_1(M)))$ becomes a covering space of $\mathcal{QH}(M)$.

Theorem 7.15 (1998). *There is a complete list of geometrically finite groups in $\mathcal{H}(M) \setminus \mathcal{QH}(M)$, which is non-minimally parabolic.*

Theorem 7.16 (Namazi–Souto).

$$\mathcal{H}(M) \subset \overline{\mathcal{QH}(M)}, \mathcal{AH}(M) = \bigcup_{N \sim M} \overline{\mathcal{QH}(N)}.$$

Here the closure is taken in the space of representations mod out conjugate. ($\mathcal{QH}(N)$'s are disjoint until the closure is taken.)

Theorem 7.17 (Sullivan). $\text{Int}\mathcal{H}(M) = \mathcal{QH}(M)$.

Geometrically infinite groups

Let $\mathcal{A}$ be a surface group. Let $\text{qf} : \mathcal{T}(S) \times \mathcal{T}(\bar{S}) \rightarrow \mathcal{QF}(S)$.

Theorem 7.18 (Bers). *For any $m_0 \in \mathcal{T}(S)$ and $n_0 \in \mathcal{T}(\bar{S})$, let*

$$B_{m_0} = \text{qf}(\{m_0\} \times \mathcal{T}(\bar{S})), \quad B'_{n_0} = \text{qf}(\mathcal{T}(S) \times \{n_0\}).$$

Take closure in $\mathcal{AH}(S)(= \mathcal{H}(S))$. Then $\overline{B_{m_0}}, \overline{B'_{n_0}}$ are compact.

Theorem 7.19. *The set of geometrically finite groups in $\overline{B_{m_0}} \setminus B_{m_0}$ has positive codimension, i.e. most of points in $\overline{B_{m_0}} \setminus B_{m_0}$ correspond to geometrically infinite groups. In particular, geometrically infinite group exists.*

Remark. Let $\varphi : S \rightarrow S$ be pseudo-Anoson (stable lamination) “generic homeomorphism”. For $(m_0, n_0) \in \mathcal{T}(S) \times \mathcal{T}(\bar{S})$, consider a sequence $\{\text{qf}(m_0, (\varphi^*)^n n_0)\}$. By Bers, it converges to some $\psi : \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{C}) \in \mathcal{AH}(S)$. As $\mathbb{H}^3/\psi(\pi_1(S))$ has a sequence of geodesics which does not stay in any compact set, ψ is minimally parabolic and geometrically infinite.

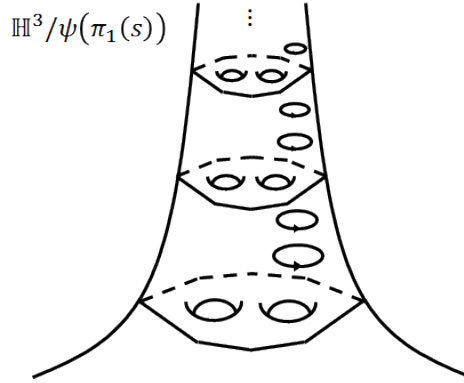


Figure 34: Convex cone in $\mathbb{H}^3/\psi(\pi_1(S))$.

Theorem 7.20 (Kleineidam–Souto, King–Lecusue?–O?). *For any M , most of points in $\overline{\mathcal{QF}(M)} \setminus \mathcal{QF}(M)$ are geometrically infinite.*

8 2018.06.29. : Thomas Koberda

Lemma 8.1 (Kopell). *A finitely generated nonabelian torsion-free group $\not\hookrightarrow \text{Diff}^2(M)$. Such groups are left orderable, $\hookrightarrow \text{Homeo}^+(M)$.*

Theorem 8.2 (Farb–Franks, Jorquera). *If G is finitely generated residually torsion-free nilpotent group, then $G \hookrightarrow \text{Diff}^1(M)$.*

Definition 8.3. Let Γ be a finite simplicial graph (i.e. at most one edge adjoins two vertices, and no loops). Define the *right-angled Artin group* (or simply RAAG) associated with Γ by

$$A(\Gamma) = \langle V \mid [v, w] = 1 \iff \{v, w\} \in E \rangle.$$

Example. 1. If Γ has no vertices, then $A(\Gamma) = \mathcal{F}(V)$ is a free group of rank $|V|$.

2. $A(K_n) = \mathcal{F}_{ab}(V)$ is a free abelian group of rank $|V|$.

Remark. RAAGs have solvable word problems.

Definition 8.4. $g = v_1^{\pm 1} \cdots v_n^{\pm 1}$ is *freely reduced* if there is no $v_i v_i^{-1}$ or $v_i^{-1} v_i$.

Theorem 8.5 (Hermiller–Meier). *$g \in A(\Gamma)$ is the identity iff it can be reduced to 1 by free reductions or switching commuting vertices.*

Exercise. Show that Dehn twist about homotopically nontrivial loops are non-trivial in $\text{Mod}(S)$.

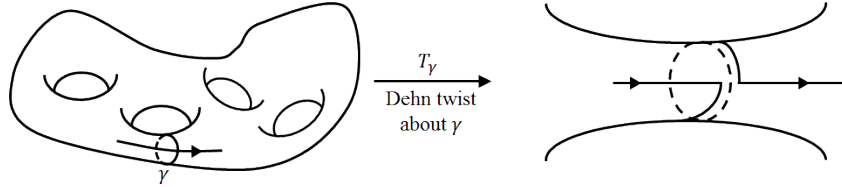


Figure 35: Dehn twist about γ .

Remark. Mapping class group contains many RAAGs. This fact relates to smoothness of the action.

Definition 8.6. Let $\gamma_1, \dots, \gamma_n$ be pairwise non-homotopic simple closed curves on S . Let $\Gamma = \Gamma(\gamma_1, \dots, \gamma_n)$, and $\{\gamma_i, \gamma_j\} \in E$ iff γ_i, γ_j have representatives in their homotopy class which are disjoint. Define $\iota_N : A(\Gamma) \rightarrow \text{Mod}(S)$ by $\gamma_j \mapsto T_{\gamma_j}^N$.

Theorem 8.7. ι_N is injective for $N \gg 0$.

Theorem 8.8 (Baik–Kim–K). *For given Γ , $A(\Gamma) \hookrightarrow \text{Diff}^\infty(\mathbb{R})$.*

Definition 8.9. An element $g \in A(\Gamma)$ is a *central word* if $g = v_1^{\pm 1} \cdots v_n^{\pm 1}$ where all these vertices commute with each other.

Proof of Theorem. The idea is to construct an interval I_w on which $A(\Gamma)$ acts nontrivially, for each $1 \neq w \in A(\Gamma)$. Then $w \notin \text{kernel}$ of the action, and it has all derivatives of all orders and bounded independent of w .

For $1 \neq w \in A(\Gamma)$, write $w = w_k \cdots w_1$ for each w_i central. Transform w into another word, a product of central words, by a left-greedy algorithm: if w_i starts with $v^\pm$ such that $[v, w_{i+1}] = 1$, then move $v^\pm$ to w_{i+1} . Let $w = w_k \cdots w_1$ be left-greedy form. We need to check that this partially defined action extends to a well-defined action of $A(\Gamma)$, and all actions can be chosen to be C^∞ . ■

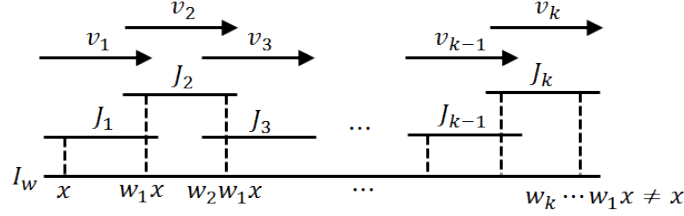


Figure 36: Left-greedy algorithm for constructing action on I_w .

Exercise. Show that the left-greedy algorithm eventually terminates.

Let $f \in \text{Homeo}^+(I)$ such that $\text{supp}(f) = \bigcup_{n \geq 1} I_n^\circ$, where $\{I_n\}$ is a collection of disjoint intervals in I . Suppose for all n , there exist $x_n \in I_n$ such that $\frac{|f(x_n) - x_n|}{|I_n|} \geq .9$. Can f be differentiable? In I_n , (say the left end-point is 0), by mean value theorem,

$$\exists y_n \in (0, x_n) \text{ s.t. } f'(y_n) = \frac{f(x_n) - x_n}{x_n} \geq 1 + \frac{.9|I_n|}{x_n} \geq 1.9.$$

Say $I_n = [a_n, b_n]$, $b_n < a_{n+1}$, and $b_n \rightarrow 1$. As $f'(y_n) \geq 1.9$ but $y_n \rightarrow 1$, contradiction.

Let $A(\Gamma)$ be RAAG. What is the largest k for which $A(\Gamma) \hookrightarrow \text{Diff}^k(M)$, where $M = I, S^1$?

Lemma 8.10 (Two-jumps Lemma, BKK). *For $f, g \in M$ continuous, let $\{y_j\}$ be a sequence of points such that*

$$I_n = [f(y_j), g(y_j)], |I_j| \rightarrow 0, y_j \in I_j^\circ.$$

Let $A_j = [f(y_j), y_j]$, $B_j = [y_j, g(y_j)]$, and assume $s_j \in \text{Fix}(g) \cap A_j$ and $t_j \in \text{Fix}(f) \cap B_j$. Then either f or g fails to be C^1 .

Proof. We may assume $y_j, s_j, t_j \rightarrow y$ such that $f(y) = g(y) = y$. Suppose f, g are C^1 . Clearly

$$f(y_j) \leq s_j < y_j < t_j \leq g(y_j).$$

Then by mean value theorem, there exist $u_j \in (s_j, y_j)$ and $v_j \in (y_j, t_j)$ such that

$$\begin{aligned} g'(u_j) &= \frac{g(y_j) - s_j}{y_j - s_j} = 1 + \frac{|B_j|}{y_j - s_j} \geq 1 + \frac{|B_j|}{|A_j|}, \\ f'(v_j) &= \frac{t_j - f(y_j)}{t_j - y_j} = 1 + \frac{|A_j|}{t_j - y_j} \geq 1 + \frac{|A_j|}{|B_j|}, \end{aligned}$$

As $f(y) = g(y) = y$, $f'(v_j), g'(u_j) \rightarrow 1$, but

$$f'(v_j)g'(u_j) \geq 2 + \frac{|B_j|}{|A_j|} + \frac{|A_j|}{|B_j|} \geq 4,$$

contradiction. ■

Exercise. After passing to a subsequence, show that $y_j, s_j, t_j \rightarrow y$ such that $f(y) = g(y) = y$.

Corollary 8.11. *Let $f, g \in \text{Homeo}^+(M)$. Suppose $\text{supp}(f) \supset \{I_j^f\}$, $\text{supp}(g) \supset \{I_j^g\}$ such that: then f or $g \notin \text{Diff}^1(M)$.*

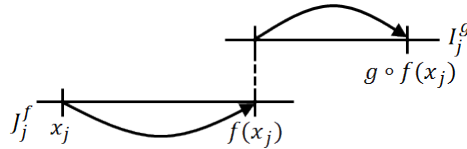


Figure 37: Assumption on the corollary.

Theorem 8.12 (BKK). $A(P_4) = \langle a, b, c, d \mid [a, b] = [b, c] = [c, d] = 1 \rangle$. Then $A(\Gamma) \not\hookrightarrow \text{Diff}^2(M)$.

Exercise. Deduce this from two-jumps lemma.

Remark. On the regularity of RAAG actions on M , all RAAGs $\hookrightarrow \text{Diff}^1(M)$, and if $A(P_4) < A(\Gamma)$, then $A(\Gamma) \not\hookrightarrow \text{Diff}^2(M)$.

Proposition 8.13. *If $\Lambda \leq \Gamma$ is a full subgraph, then $A(\Lambda) \leq A(\Gamma)$.*

Theorem 8.14 (Kim–K). *$A(P_4) \leq A(\Gamma)$ iff $P_4 \leq \Gamma$ is a full subgraph.*

Definition 8.15. A P_4 -free graph, meaning that P_4 is not a full subgraph, is called a *cograph*.

Remark. The class of cographs contains a singleton vertex, is closed under finite disjoint unions and finite joins. Thus RAAGs on cographs contains $\mathbb{Z}$, and is closed under finite free product and finite direct product.

Theorem 8.16 (KK). *If $A(\Gamma) \hookrightarrow \text{Diff}^2(M)$ then*

$$A(\Gamma) \simeq A(\Gamma_1) \times \cdots \times A(\Gamma_k), \quad A(\Gamma_i) \simeq \mathbb{Z}^{n_{i,1}} * \cdots * \mathbb{Z}^{n_{i,j_i}}.$$

If $A(\Gamma)$ is of the above form, then actually $A(\Gamma) \hookrightarrow \text{Diff}^\infty(M)$.

Theorem 8.17 (Nielsen, Handel–Thurston). $\text{Mod}_{g,1} \hookrightarrow \text{Homeo}^+(S^1)$.

Theorem 8.18 (Farb–Franks, Ghys). *If $g \geq 3$, then $\text{Mod}_{g,1} \not\hookrightarrow \text{Diff}^2(M)$.*

Theorem 8.19 (Parwani). *If $g \geq 6$, then $\text{Mod}_{g,1} \not\hookrightarrow \text{Diff}^1(M)$.*

On the regularity of finite index subgroups of $\text{Mod}_{g,1}$ on M ?

Definition 8.20. For $S_{g,n,b}$, $c(S) = 3g - 3 + n + b$.

Theorem 8.21 (BKK). *If $\text{Mod}_{g,n,b}$ has a finite index subgroup $G \hookrightarrow \text{Diff}^2(M)$, then $c(S) \leq 1$, i.e. S is either $S_{0,\leq 4}$ or $S_{1,\leq 1}$. The converse also holds.*

Exercise. Prove this from $A(\Gamma) \hookrightarrow \text{Mod}(S)$ but $A(P_4) \not\hookrightarrow \text{Diff}^2(M)$.

Definition 8.22. For $\alpha \in [1, \infty)$, let $\alpha = k + \epsilon$, $k = \lfloor \alpha \rfloor$. Then $\text{Diff}^\alpha(M)$ is the set of C^k -diffeomorphisms whose k^{th} derivatives are ϵ -Hölder continuous. (What about Lipschitz continuous?)

Definition 8.23. *Critical regularity* of a countable subgroup $G < \text{Homeo}^+(M)$ is defined by

$$\text{CR}(G) = \sup_{\alpha} \{G \hookrightarrow \text{Diff}^\alpha(M)\}.$$

Theorem 8.24 (Jorquera–Navas–Rivas). *Let $H = \langle x, y, z \mid z = [x, y] = 1 \rangle$ be the Heisenberg group. Then $\text{CR}(H) = 2$. Let N_4 be the set of upper-triangular 4×4 matrix whose diagonal entries are 1. Then $\text{CR}(N_4) = 1.5$.*

Remark. Critical regularity may not be achieved.

Definition 8.25. $\mathcal{G}^\alpha$ is the set of countable subgroups of $\text{Diff}^\alpha(M)$, up to isomorphism.

Remark. $\mathcal{G}^\alpha \supset \mathcal{G}^\beta$ if $\beta \geq \alpha$.

Theorem 8.26 (KK). *1. $\mathcal{G}^\alpha \setminus \bigcup_{\beta > \alpha} \mathcal{G}^\beta$ contains a finitely generated group and (non-finitely generated) simple group.*

2. $\bigcap_{\beta < \alpha} \mathcal{G}^\beta \setminus \mathcal{G}^\alpha$ contains a finitely generated group and (non-finitely generated) simple group. (actually, both contain uncountably many such groups.)

9 2018.06.29. : Koji Fujiwara

Proposition 9.1. *Suppose G acts properly on a δ -hyperbolic space X . Assume there exists $g \in G$ hyperbolic on X with an axis γ . Then there exists $\xi > 0$ such that for every $a \in G$, either $|\pi_\gamma(a\gamma)| \leq \xi$ or $\gamma \sim a\gamma$. Call this property by (P0). (projection uniformly bounded.)*

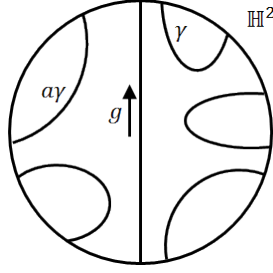


Figure 38: Description of (P0).

Let $\mathbb{Y} = \{a\gamma \mid a \in G\} / \sim$. We still write $\pi_\gamma(a\gamma)$ after $\sim$ instead of $\pi_{[\gamma]}([a\gamma])$. Then by the above proposition, there exists $\xi > 0$ such that for every $A \neq B \in \mathbb{Y}$, $|\pi_A(B)| \leq \xi$. Indeed, if $A = a\gamma$, $B = b\gamma$ for some $a, b \in G$, then $\gamma \not\sim a^{-1}b\gamma$ so $|\pi_\gamma(a^{-1}b\gamma)| \leq \xi$. As a defines an isometry on X , $|\pi_{a\gamma}(b\gamma)| \leq \xi$.

Note. $\pi_A(B)$ is G -equivalent, i.e. for any $g \in G$, $\pi_{gA}(gB) = g \cdot \pi_A(B)$.

Definition 9.2. For distinct $A, B, C \in \mathbb{Y}$,

$$d_A(B, C) = \text{diam}_A(\pi_A(B) \cup \pi_A(C)).$$

Proposition 9.3 (Behrstuhl? inequality). *There exists $\xi > 0$ such that for every distinct $A, B, C \in \mathbb{Y}$, if $d_A(B, C) \geq \xi$, then $d_B(A, C), d_C(A, B) \leq \xi$. Call this property by (P1).*

Exercise. Prove the above proposition using the below figure.

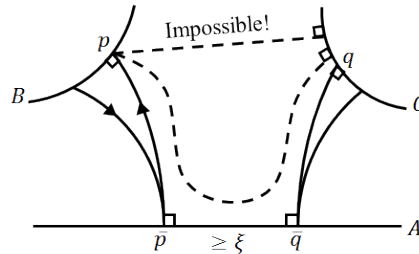


Figure 39: Description of (P1).

Proposition 9.4. *There exists $\xi > 0$ such that for any distinct $A, C \in \mathbb{Y}$, $\{B \in \mathbb{Y} \mid d_B(A, C) \geq \xi\}$ is finite. Call this property by (P2). (finitely many projections.)*

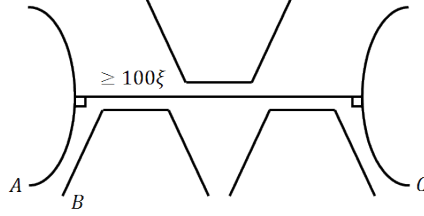


Figure 40: Description of (P2).

Note. If $d_B(A, C) \geq \xi$, then B stays close to the shortest segment between A, C .

Theorem 9.5 (BBF). *Suppose $G \curvearrowright X$ geodesic space, and let $\mathbb{Y}$ be G -invariant collection of subsets of X such that each $Y \in \mathbb{Y}$ is (uniformly) quasi-isometric to $\mathbb{R}$. Let $\pi_Y : X \rightarrow Y$ be the nearest points projection for $Y \in \mathbb{Y}$, and define $d_Y(X, Z) = \text{diam}(\pi_Y(X), \pi_Y(Z))$.*

Assume there exists $\xi > 0$ such that the above (P0), (P1), and (P2) hold. Then there is a quasi-tree Q on which G acts such that: $\bigsqcup_{Y \in \mathbb{Y}} Y \hookrightarrow Q$ G -equivalent; and for each $K \gg \xi$, for all $X, Z \in \mathbb{Y}$, we have the discrete formula:

$$d_Q(x, z) \geq \frac{1}{4} \sum_{Y \in \mathbb{Y}} d_Y(x, z)_K, \quad x \in X, z \in Z.$$

Here,

$$d_Y(x, z)_K = \begin{cases} 0, & \text{if } d_Y(x, z) \leq K, \\ d_Y(x, z), & \text{otherwise,} \end{cases}$$

and $d_Y(x, z) = \text{diam}(\pi_Y(x) \cup \pi_Y(z))$.

Remark. For the construction of Q , fix $K \gg \xi$. We want to add edges to $\mathbb{Y}$. Join $A, C \in \mathbb{Y}$ by an edge at $\pi_A(C)$ and $\pi_C(A)$ if $\{B \in \mathbb{Y} \mid d_B(A, C) > K\} = \emptyset$. If $d_B(A, C) > K$, we say B is *between* A and C . This constructs Q .

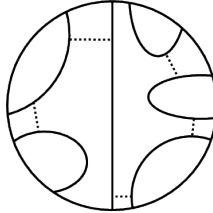


Figure 41: Construction of Q .

Example. $\mathbb{F}_2 = \langle a, b \rangle \curvearrowright T$, where T is the Cayley graph for $\mathbb{F}_2$. Let $\alpha \subset T$ be an axis for a , and let $\mathbb{Y} = \mathbb{F}_2 \cdot \alpha$ be the set of horizontal lines. For instance, we do not join α , $a^2b^2a^3b\alpha$ but join α , $b^{-1}\alpha$.

Theorem 9.6 (BBF). $\text{MCG}(\Sigma) \curvearrowright X = X_1 \times \cdots \times X_n$, where X_i are quasi-trees. Moreover, $\text{MCG}(\Sigma) \hookrightarrow X$ by quasi-isometrics.

Definition 9.7. A simple closed curve α on S is *essential* if it is neither null-homotopic nor homotopic to ∂S .

Definition 9.8. Let $S_{g,p}$ be a compact orientable surface of genus g with p boundary components. The *curve graph* $\mathcal{C}(S)$ of S is a graph whose vertices are homotopy classes of essential simple closed curves on S . We join distinct α, β if $\alpha \cap \beta = \emptyset$ after homotopy.

Definition 9.9. $S_{g,p}$ is sporadic if $g = 0$ and $p \leq 4$, or $g = 1$ and $p \leq 1$.

Remark. $\text{MCG}(\Sigma) \curvearrowright \mathcal{C}(S)$ by automorphisms.

Definition 9.10. The *intersection number* $I(\alpha, \beta)$ is defined by

$$I(\alpha, \beta) = \min_{\text{homotopy}} \#(\alpha \cap \beta).$$

Lemma 9.11. Assume S is not sporadic. Then for all $\alpha, \beta \in \mathcal{C}(S)$,

$$d_{\mathcal{C}(S)}(\alpha, \beta) \leq 2I(\alpha, \beta) + 1.$$

In particular, $\mathcal{C}(S)$ is connected.

Proof. Induction on $I(\alpha, \beta)$. If $I(\alpha, \beta) = 0$, $d_{\mathcal{C}(S)}(\alpha, \beta) = 0$. If $I(\alpha, \beta) = 1$, then $\partial N_\epsilon(\alpha \cup \beta)$ is essential, because otherwise $\partial N_\epsilon(\alpha \cup \beta)$ bounds a disk or a boundary curve. Thus $\partial N_\epsilon(\alpha \cup \beta)$ is disjoint from α, β , and $d_{\mathcal{C}(S)}(\alpha, \beta) \leq 2$. Let us show $d_{\mathcal{C}(S)}(\alpha, \beta) - 2I(\alpha, \beta) \leq 1$ for $I(\alpha, \beta) \geq 2$. We do “surgery” to β :

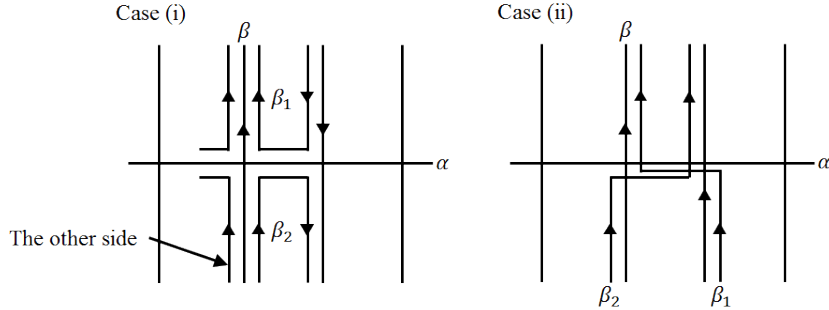


Figure 42: Surgery to β .

For the case (i), construct simply closed curves β_1, β_2 as in the figure, then $I(\alpha, \beta_i) \leq I(\alpha, \beta) - 2$ and $d(\beta, \beta_i) \leq 1$. Hence

$$d(\alpha, \beta) \leq d(\alpha, \beta_i) + d(\beta_i, \beta) \leq d(\alpha, \beta_i) + 1 \leq 2I(\alpha, \beta_i) + 2 \leq 2I(\alpha, \beta) - 2.$$

So if β_1 or β_2 are essential, done. If not, use the other side. For the case (ii), $I(\alpha, \beta_i) \leq I(\alpha, \beta) - 1$ and $I(\beta, \beta_i) \leq 1$, so $d(\beta, \beta_i) \leq 2$. As β_i is essential, $d(\alpha, \beta) \leq d(\alpha, \beta_i) + 1$. ■

Definition 9.12. A subsurface A of S is *incompressible* if every component of ∂A is essential in S .

Let $\mathbb{Y}$ be the set of homotopy classes of incompressible, proper, connected subsurfaces A in $S_{g,p}$. Assume for any $A \neq B \in \mathbb{Y}$, $\partial A \cap \partial B \neq \emptyset$, i.e. $I(\partial A, \partial B) \neq 0$. For any $Y \neq A \in \mathbb{Y}$, define

$$\pi_Y(A) = \partial A|_Y \subset \mathcal{CA}(Y),$$

where $\mathcal{CA}(Y)$ is the *curve and arc graph* of Y , of which vertices are homotopy classes of essential simple connected on Y and essential arcs on $(Y, \partial Y)$, and join them if disjoint after homotopy. Note that $\mathcal{C}(Y) \hookrightarrow \mathcal{CA}(Y)$ by quasi-isometrics.

Proposition 9.13. (P0) $\dim \pi_Y(A) \leq 1$ in $\mathcal{CA}(Y)$, and

$$d_Y(A, B) = d_{\mathcal{C}(Y)}(\partial A|_Y, \partial B|_Y).$$

Proposition 9.14. (P1) If $d_X(Y, Z) \geq 10$, then $d_Y(X, Z), d_Z(X, Y) \leq 3$.

Proposition 9.15. (P2) For any $X \neq Y \in \mathbb{Y}$, $\#\{Z \in \mathbb{Y} \mid d_Z(X, Y) \geq 4\}$ is finite.

Proof of (P1). Suppose $d_X(Y, Z) \geq 10$. Then ∂Y and ∂Z intersect at least 3 points in X . Then there exists $\sigma \in \partial Z \cap Y \cap X$ but σ is distinct from ∂X , so

$$d_Y(\partial X|_Y, \partial Z|_Y) = d_Y(\partial X|_Y, \sigma) \leq 1 + 1 + 1 = 3.$$

■

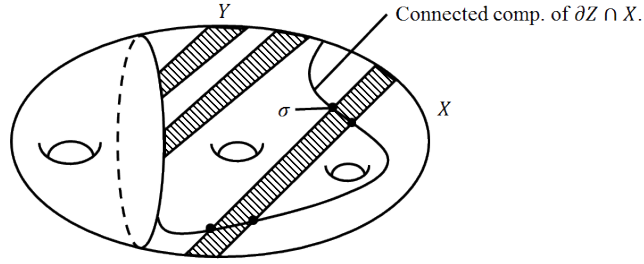


Figure 43: X is the right part, and Y is the blackened part.

By applying the theorem by BBF, there exists a product X of quasi-trees such that $\varphi : \coprod_{Y \in \mathbb{Y}} Y \hookrightarrow X$. Moreover, if there exists $H < \text{MCG}(\Sigma)$ which acts on $\mathbb{Y}$, then H acts on X and φ is H -equivalent.

Theorem 9.16 (Masur–Minsky). $\mathcal{C}(S)$ is δ -hyperbolic.

10 2018.07.02. : Mladen Bestvina

CLASSIFICATION OF AUTOMORPHISMS.

Example. 1. $\Phi : a \mapsto a, b \mapsto ab$ is parabolic.

2. $\Phi : a \mapsto b, b \mapsto ab$ is hyperbolic, $D_\Phi = \log \frac{1+\sqrt{5}}{2}$.

Proof. Take $\Gamma = R$ with the identity mapping. Let $f : R \rightarrow \Gamma$ given by $a \mapsto b, b \mapsto ab$ be a marking. We first assign a metric l to Γ such that f expands each edge by the same factor λ :

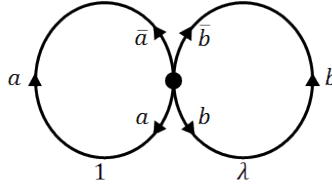


Figure 44: Example - D_Φ is realized.

$l(a) = 1, l(b) = l(f(a)) = \lambda$, so $1 + \lambda = l(ab) = l(f(b)) = \lambda^2$, hence $\lambda = \frac{1+\sqrt{5}}{2}$. Hence $d(\Gamma, \Gamma\Phi) \leq \log \lambda$, $D_\Phi \leq \log \lambda$. Second, we find a train-track structure of Γ preserving by f , in the sense that f sends legal paths to legal paths. Define $d \sim d' \iff f_*^k(d) = f_*^k(d')$ for large k . Clearly $\bar{a} \sim \bar{b}$. To see that f sends legal paths to legal paths, it suffices to check that (i) the image of any edge by f is legal, and (ii) the images of distinct gates by f_* are distinct gates. Note that the condition (i) is stable under taking positive powers, so $d(\Gamma, \Gamma\Phi^k) = \log \lambda^k = k \log \lambda$. Suppose $\Gamma' \in \mathcal{CV}_2$, and $D = d(\Gamma', \Gamma'\Phi) < \log \lambda$. Then there is a difference of marking $f' : \Gamma' \rightarrow \Gamma'\Phi = \Gamma'$ with slope of $f' < \lambda$. Then

$$\begin{aligned} k \log \lambda = d(\Gamma, \Gamma\Phi^k) &\leq d(\Gamma, \Gamma') + d(\Gamma', \Gamma'\Phi^k) + d(\Gamma'\Phi^k, \Gamma\Phi^k) \\ &\leq d(\Gamma, \Gamma') + d(\Gamma', \Gamma) + kD, \end{aligned}$$

so

$$k(\log \lambda - D) \leq d(\Gamma, \Gamma') + d(\Gamma', \Gamma),$$

which is not true. ■

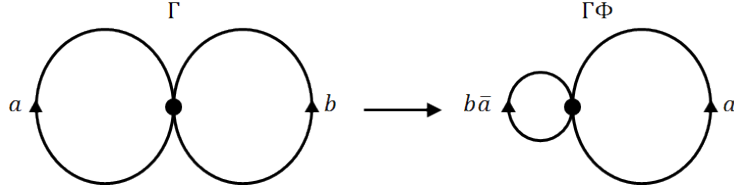


Figure 45: Example - the group action on Γ .

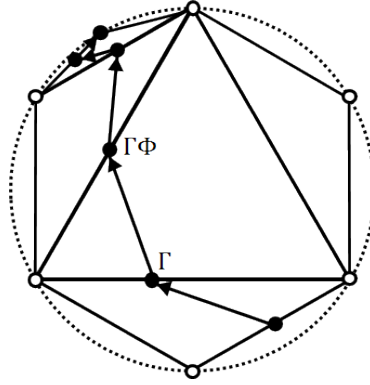


Figure 46: Example - the orbit of Φ in $\mathcal{CV}_2$. ($\text{Out}(\mathbb{F}_2) \simeq \text{GL}_2(\mathbb{Z})$)

Definition 10.1. For a hyperbolic automorphism Φ , define $\text{Munset}(\Phi) = \{\Gamma \mid d(\Gamma, \Gamma\Phi) = D_\Phi\}$.

Theorem 10.2. Suppose $\Phi \in \text{Out}(\mathbb{F}_n)$ is represented by a self-map $f : \Gamma \rightarrow \Gamma$, where Γ is a marked graph, and there is a f -invariant subgraph $\Delta \subseteq \Gamma$ and a metric l on Γ such that:

1. The slope of f on any edge in Δ is $\lambda > 1$,
2. The slope of f on any edge in $\Gamma - \Delta$ is $< \lambda$,
3. There is a non-degenerate (i.e. 2 gates at every vertex) train-track structure on Δ such that f sends legal paths to legal paths.

Then Φ is hyperbolic and $D_\Phi = \log \lambda$, $\Gamma \in \text{Munset}(\Phi)$.

Example. $a \mapsto b$, $b \mapsto ab$, and $c \mapsto cab\bar{a}\bar{b}$. Set $\Delta = a \cup b$. On a, b same metric, and on before c much larger. Then by the above theorem, Φ is hyperbolic.

Theorem 10.3 (Classification). 1. (Nielsen realization) An automorphism is elliptic iff it has a finite order.

2. Every hyperbolic automorphism Φ admits a representative as in the above theorem. In fact, every $\Gamma \in \text{Munset}(\Phi)$ can be approximated by such a representative. It can also be arranged that $f : \Gamma \rightarrow \Gamma$ sends all vertices to vertices and all vertices have valence ≥ 3 . Moreover, the maximal slope is an algebraic integer λ . ($D_\Phi = \log \lambda$.)

3. Every parabolic automorphism is reducible: it admits a representative $f : \Gamma \rightarrow \Gamma$ such that there is a proper f -invariant subgraph $\Delta \neq \emptyset$ that has no contractible components.

Proposition 10.4. If Φ is hyperbolic and $D_\Phi = \log \lambda$, then there is a constant $C \geq 1$ such that:

1. for every conjugacy class α , $|\Phi^k(\alpha)| \leq C \cdot \lambda^k \cdot |\alpha|$, where $|\cdot|$ is the length of cyclically reduced word.
2. there exists a nontrivial conjugacy class α such that $|\Phi^k(\alpha)| \geq \frac{1}{C} \cdot \lambda^k \cdot |\alpha|$.

Example. Set $\Phi : a \mapsto b, b \mapsto ab, c \mapsto da, d \mapsto cd$. Set a graph Γ_M as in the below figure for large M , then the slope $\rightarrow \lambda$ as $M \rightarrow \infty$. Then $D_\Phi \leq \log \lambda$, $D_\Phi \geq \log \lambda$ by looking at a, b -subgraph. We claim that Φ is parabolic. Indeed, if Φ is hyperbolic, (1) from the above proposition holds. But $|\Phi^k(\alpha)| \sim k \cdot \lambda^k$.

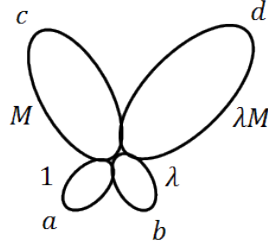


Figure 47: Example - the “approximate” graph.

Proof of theorem-(2). Let $\Gamma \in \text{Munset}(\Phi)$. We will approximate Γ by a representative as above. Let $f : \Gamma \rightarrow \Gamma$ be a representative on Γ , whose largest slope is λ , i.e. $D_\Phi = \log \lambda$. After approximating as above, we may assume that Γ has the tension graph $\Delta_f \subset \Gamma$ that has a non-degenerate train-track structure.

First, we arrange $f(\Delta) \subseteq \Delta$. Suppose e is an edge in Δ and $f(e) \not\subseteq \Delta$. Scale the metric on Γ : up on Δ , down on $\Gamma - \Delta$ a tiny, so that the slope on every edge is still $\leq \lambda$, and it is $< \lambda$ in $\Gamma - \Delta$. It is also $< \lambda$ on e , so by deleting e , the tension graph decreases. If vertices with 1 gate appear on the new Δ , go back to “clean up”. This decreases Δ further. Repeat these operations until we arrive at a representative $f : \Gamma \rightarrow \Gamma$, where the tension graph Δ has no 1-gate part and $f(\Delta) \subseteq \Delta$.

Second, there arranges that f preserves the train-track structure. This will require changing the topology of Γ . Say $e \subset \Delta$ is an edge, and $f(e) \subset \Delta$ is not legal. Change Γ by identifying small intervals of same length in the directions d, d' . Let $\Gamma' = \Gamma / \sim$, and

$$\begin{array}{ccc} \Gamma & \longrightarrow & \Gamma' \\ f \downarrow & & \downarrow f' \\ \Gamma & \longrightarrow & \Gamma' \end{array}$$

So f' has slope λ on the image Δ' of Δ and slope $< \lambda$ on other edges. But f' backtracks on the image of e , so after a homotopy, the tension graph decreases. Say d, d' are inequivalent directions at a vertex of Δ , but $f_*(d) * f_*(d')$. By folding $f_*(d)$ and $f_*(d')$, d and d' become equivalent. ■

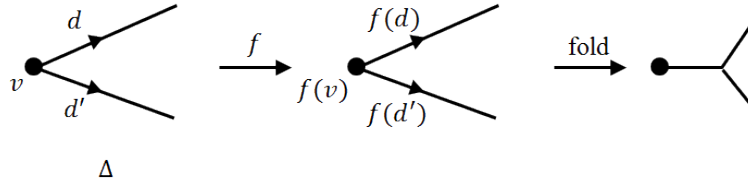


Figure 48: Description of the proof.

There is a notion of complexity that always decreases: complexity of Δ , complexity of the train-track structure, and $\#$ of edges in Δ .

Definition 10.5. A *core graph* is a graph hanging no subtrees.

Definition 10.6. *Complexity* of core graph Γ is (rank of $H_1(\Gamma)$, - $\#$ components).

Definition 10.7. *Complexity* of non-degenerate train-track structure is

$$\sum_{v \text{ vertex}} (\# \text{ gates at } v) - 2.$$

Definition 10.8. In Riemannian manifold M , $\text{syst}(M)$ is the length of a shortest essential loop in M .

Proof of theorem-(3). Suppose $D_\Phi \geq 0$ is not realized. Then there is a sequence $\{\Gamma_i\}$ in $\mathcal{CV}_n$ such that $d(\Gamma_i, \Gamma_i\Phi) \rightarrow D_\Phi$. There are two cases to consider:

Case 1. The sequence $\{\Gamma_i\}$ “leaves every thick part” of $\mathcal{CV}_n$. Then $\text{syst}(\Gamma_i) \rightarrow 0$ as $i \rightarrow \infty$. Let λ_i be the largest slope for $f_i : \Gamma_i \rightarrow \Gamma_i$, then $\lambda_i \rightarrow \lambda$, so $\lambda_i \leq C$ for all i . As f_i should send small loop to small loop for large i , small loop is f_i -invariant subgraph. For $\Gamma \in \mathcal{CV}_n$ and $\epsilon > 0$, define $\Gamma(\epsilon)$ to be the union of essential loops (not necessarily immersed) of length $\leq \epsilon$. Let $\widehat{\Gamma}(\epsilon)$ be the associated core graph, by removing hanging trees from $\Gamma(\epsilon)$. If $\epsilon < \frac{1}{3n-3}$ then $\Gamma(\epsilon) \subsetneq \Gamma$. Choose $\frac{1}{3n-3} > \epsilon > \frac{\epsilon}{C} > \dots > \frac{\epsilon}{C^k}$, where k is the length of a largest chain of subgraphs in a rank n graph ($k = 3n - 3$). Then

$$\widehat{\Gamma}_i(\epsilon) \supseteq \widehat{\Gamma}_i\left(\frac{\epsilon}{C}\right) \supseteq \dots \supseteq \widehat{\Gamma}_i\left(\frac{\epsilon}{C^k}\right) \neq \emptyset,$$

for large i . So there exists j such that $\widehat{\Gamma}_i(\frac{\epsilon}{C^j}) = \widehat{\Gamma}_i(\frac{\epsilon}{C^{j+1}})$. If $f_i : \Gamma_i \rightarrow \Gamma_i$ has slope $\leq C$, then it maps loops of length $\leq \frac{\epsilon}{C^{j+1}}$ to loops of length $\leq \frac{\epsilon}{C^j}$. Thus

$$f\left(\widehat{\Gamma}_i\left(\frac{\epsilon}{C^{j+1}}\right)\right) \subseteq f\left(\Gamma_i\left(\frac{\epsilon}{C^{j+1}}\right)\right) \subseteq \Gamma_i\left(\frac{\epsilon}{C^j}\right) = \widehat{\Gamma}_i\left(\frac{\epsilon}{C^j}\right) \cup (\text{hanging trees}).$$

After compositing f with a deformation that retracts $\Gamma_i(\frac{\epsilon}{C^j})$ to $\widehat{\Gamma}_i(\frac{\epsilon}{C^j})$, we have an f -invariant graph.

Case 2. $\{\Gamma_i\}$ “stays in a thick part”. Then $\text{syst}(\Gamma_i) \geq \epsilon > 0$ for all i . Here we argue that Φ is hyperbolic. The idea is to choose forest(?) markings for the moment. After a subsequence, all Γ_i are homeomorphic and can be represented by paints in a simplex. (i.e. after a subsequence $\Gamma_i \rightarrow \Gamma$.) We have a representation $f_i : \Gamma_i \rightarrow \Gamma_i$ of slope $\lambda_i \rightarrow \lambda$ ($D_\Phi = \log \lambda$). Then $h_i : \Gamma \rightarrow \Gamma_i$, $g_i : \Gamma_i \rightarrow \Gamma$ are difference of markings of slope $\rightarrow 1$, so

$$\begin{array}{ccc} \Gamma_i & \xrightarrow{f_i} & \Gamma_i \\ h_i \uparrow & & \downarrow g_i \\ \Gamma & \xrightarrow{g_i f_i h_i} & \Gamma \end{array}$$

with slope of $g_i f_i h_i \rightarrow \lambda$. Hence by Arzela–Ascoli theorem, there is a subsequence $g_i f_i h_i \rightarrow F : \Gamma \rightarrow \Gamma$ with slope λ . Hence Φ is hyperbolic. Contradiction. ■

11 Problems : Mladen Bestvina

11.1 Folding and applications

1. Find a basis of the subgroup

$$H = \langle \bar{b}\bar{a}b\bar{a}a\bar{b}a, ab\bar{a}b\bar{a}b\bar{a}, ab\bar{a}b\bar{a}b\bar{a} \rangle < \langle a, b \rangle.$$

2. Given $w \in \mathbb{F}_n$ give an algorithm to decide whether $w \in H$. E.g. show $a \notin H$ of #1.
3. Given $w \in \mathbb{F}_n$ give an algorithm to decide whether w is conjugate into H .
4. Can you tell if H is normal in $\mathbb{F}_n$?
5. Can you tell if H has finite index in $\mathbb{F}_n$?
6. Suppose H is a finitely generated subgroup of $\mathbb{F}_n$. Show that either H has finite index in $\mathbb{F}_n$ or $H = 1$.
7. Given a homomorphism $h : \mathbb{F}_n \rightarrow \mathbb{F}_m$, can you tell if h is injective, surjective, bijective? In particular, show that $\mathbb{F}_n$ is Hopfian, i.e. every epimorphism $\mathbb{F}_n \rightarrow \mathbb{F}_n$ is an automorphism. [Answer: Injective iff there are no folds of the second kind. Surjective iff the last map is a homeomorphism.]
8. Let $h : \langle a, b \rangle \rightarrow \langle a, b \rangle$ be given by $h(a) = ab\bar{b}ab$, $h(b) = bab\bar{a}bab$. Show that h is an automorphism and compute h^{-1} . (You can do this by messing about. But try to do it algorithmically, that is, decompose h into a product of Nielsen generators and then compose the inverses in opposite order. The point is that this can be programmed on a computer.)
9. Show that for every homomorphism $h : \mathbb{F}_n \rightarrow \mathbb{F}_m$ there is a free factorization $\mathbb{F}_n = A * B$ such that h kills A and is injective on B .
10. (Marshall Hall's theorem) Show that for every finitely generated $H < \mathbb{F}_n$ there is a subgroup $H' < \mathbb{F}_n$ such that $H < H'$, H is a free factor in H' , and H' has finite index in $\mathbb{F}_n$. You can find H' algorithmically. Do it for H in the example from #1.
11. Let $H < \mathbb{F}_n$ be finitely generated. Can you always compute the normalizer $N(H)$? What can you say about the index $[N(H) : H]$? E.g. show that $N(H) = H$ for H as in #1. [Answer: It is always finite and bounded by the number of vertices in the graph representing H . Recall that $N(H)/H$ is the deck group.]

12. If T and T' are two maximal trees, show that there is a sequence $T = T_0, T_1, \dots, T_k = T'$ of maximal trees such that any two consecutive trees differ in only one edge, as in the lecture.
13. *This is a bit more ambitious. Consider the simplicial complex whose vertices are non-closed edges of G , and a collection of edges spans a simplex if their union is a forest. Draw some examples. Can you make a conjecture about the homotopy type of the complex?
14. *Read the wonderful paper *Topology of finite graphs* by John Stallings. (Inventiones 71 (1983) 551-565.)

11.2 Outer space

15. Consider a graph with two vertices and four edges, all joining the two vertices. Once a marking is provided, this graph defines a simplex with missing faces in Outer space $\mathcal{CV}_3$. How many faces are missing? How many simplices-with-missing-faces in $\mathcal{CV}_3$ contain this simplex?
16. In $\mathcal{CV}_2$, sketch an orbit of the automorphism $a \mapsto a, b \mapsto ab$, and also of $a \mapsto b, b \mapsto ab$.
17. Prove that $\mathbb{Z}^m$ cannot act freely and properly discontinuously on a contractible complex of dimension $< m$. Deduce that no subgroup of $\text{Out}(\mathbb{F}_n)$ is isomorphic to $\mathbb{Z}^{2n-2}$ ($n \geq 2$) (recall that the dimension of the spine is $2n - 3$). Find a subgroup isomorphic to $\mathbb{Z}^{2n-3}$.
18. The smallest dimension of a $\Sigma(\Gamma)$ is $n - 1$. (Recall that $\Sigma(\Gamma)$ is the space of normalized non-degenerate metrics on the graph Γ of rank n .)
19. The largest dimension of a $\Sigma(\Gamma)$ is $3n - 4$.
20. $\text{Out}(\mathbb{F}_n)$ has finitely many conjugacy classes of finite subgroups. (Use Nielsen realization.)
21. Find a nontrivial element of finite order in the kernel of $\text{Out}(\mathbb{F}_n) \rightarrow \text{GL}_n(\mathbb{Z}/2)$. Show that every such element has order 2 and that therefore every finite subgroup of the kernel is abelian (in fact, a direct sum of $\mathbb{Z}/2$'s). Can you find the largest such subgroup?
22. Can you find estimates on size of the largest finite subgroup of $\text{Out}(\mathbb{F}_n)$? For example, the stabilizer of a rose has order $2^n n!$. Can you find a larger finite group? What about $n = 2$ and 3 ? For the answer see Wang-Zimmermann: *The maximum order of finite groups of outer automorphisms of free groups*. Math Z., 216, 1994, 83-87.

11.3 Lipschitz metric and train tracks

23. For $\epsilon > 0$ find example of graphs Γ, Γ' such that $d(\Gamma, \Gamma') < \epsilon$ and $d(\Gamma', \Gamma) > 1/\epsilon$.
24. Let $R \in \mathcal{CV}_2$ be the rose with identity marking and edges of length $1/2$. Let f be given by $a \mapsto a, b \mapsto ab$. Show that $d(R, f^k(R)) \sim \log k$.
25. For the same R and f , show that $d(R, f^k(R))$ is bounded above and below by a linear function of k . In fact, if R is replaced by a suitable graph in the same simplex, $k \mapsto d(R, f^k(R))$ is a linear function on the rose. [Hint: Use the train track metric. By connecting consecutive graphs in the orbit with a folding path one gets an *axis* of f .]
26. For f in #25, show that axes for f and for f^{-1} are distinct lines.
27. For f in #24 find a sequence Γ_k with $d(\Gamma_k, f(\Gamma_k)) \rightarrow 0$. Prove that for any such sequence for large k there is going to be a proper invariant subgraph (up to homotopy).
28. Show that in any graph with a train track structure with at least two gates at every vertex, there is a legal loop that is either embedded, or it forms a “figure 8” crossing each edge once, or it forms a “dumbbell”, crossing edges in the two loops once and edges in the connecting arc twice. See the following Figure.

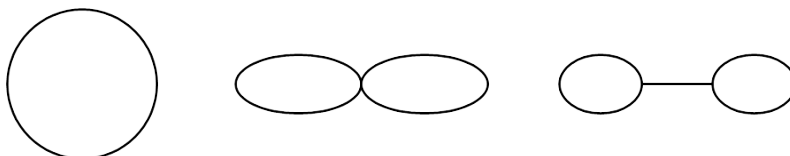


Figure 49: Possible forms of candidates. Train track structure is suggested by the pictures.

We say that an immersed loop in a graph Γ (without any train track structure) is a *candidate* if it has form as in Exercise #28.

29. Let R_3 be the rose in $\mathcal{X}_3$ with all edges of length $\frac{1}{3}$ and with inverse marking given by a, b, c , and let Γ be another such rose but with inverse marking given by $ab\bar{a}, bac\bar{b}, a$. Find all candidates in each that are witnesses for the distance to the other.
30. Consider the automorphism Φ of $\mathbb{F}_4 = \langle a, b, c, d \rangle$ given by $a \rightarrow b \rightarrow c \rightarrow d \rightarrow \bar{a}\bar{d}\bar{c}\bar{b}$.

- (a) Let R be the rose with the identity marking (so the edges correspond to a, b, c, d) and with all lengths $\frac{1}{4}$. Compute $d(R, R\Phi)$.
- (b) Find the graph Γ in the same simplex as R (i.e. the same marking, but edge lengths can be arbitrary) so that $d(\Gamma, \Gamma\Phi)$ is minimal.
- (c) Can you find a graph Γ' in a small neighborhood of Γ such that $d(\Gamma', \Gamma'\Phi) < d(\Gamma, \Gamma\Phi)$?