

Mar 9 - 2021.

PH - partially hyperbolic diffeomorphism on M^n

E^s, E^c, E^u

Stable $\downarrow$ unstable

center = in between

Aspects

topological/classification, ergodic theory.

E^s, E^u are always uniquely integrable

E^c ?? Hölder continuous -

not always integrable

in dim 3 - E^c has dimension 1

there are always center curves ($\parallel E^c$)

through every point.

DC - dynamically coherent - f invariant foliations
tangent to E^s, E^u

Example of non DC in T^3 (HHU)

$\hookrightarrow \exists$ trans $\parallel E^u$ another $\parallel E^s$

Pujals

Conjecture (Pujals 2001 question, Bonatti-Wilkinson 2005)

Let f PH transitive on M^3 . Then f is dynamically
coherent and up to finite iterates and leaf

conjugacy f is one of the 3 examples.

1) linear Anosov T^3 , 2) skew product, 3) time one
map Anosov flow

Leaf conjugacy - this assumes DC so there
are center foliations. f, g DC are leaf conjugate

if there is homeo h sending center foliation to center foliation and conjugating the action of $f|_g$.

The conjecture has been disproved - through many examples.

Theorem (Bonatti-Gogolev-Potrie) ⁽²⁰¹⁶⁾ There are examples

of f PH so that

- M admits an Anosov flow
- f^n not homotopic to identity $\forall n > 0$
- f is volume preserving.
- f is robustly transitive + stable ergodic.

why does this prove Puzos conjecture?

First property (Anosov flows) implies $\pi_1(M)$ has exponential growth (Plante-Thurston) (~~2~~, ~~2~~)

second property implies no iterate homotopic to id so not leaf conjugate to $\mathbb{Z}$. 3rd property implies transitive.

we explain 2 bet examples in $M = T^1 S$

induced by Dehn twist in S

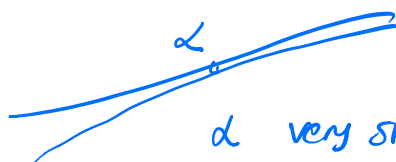
S closed
hyp surface

Assume

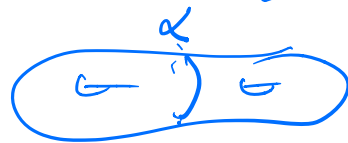
simple

It geodesic flow orientable

S has a closed geodesic of extremely small length



α very short.



we consider derivative of Dehn twist

$u \propto 1, D\alpha = h$ (take unit vectors and unit vector a scalar of image)
 proposed f is $f = \Phi_{t_1} \circ h \circ \Phi_{t_0}$ $t_0, t_1 \gg 0$???

f disturbs a lot E^S contracted
 E^u expanded.

Aside one can prove PH using cones.
 Cone about a subbundle

$\cup \supset$ cone at every point
 varies continuously.



PH can be described using cones. and properties of cones $g(\theta^u)$

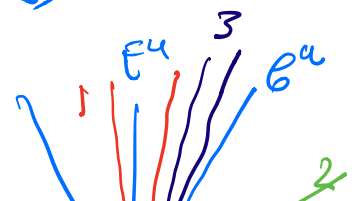
Other containment properties for $\theta^S, \theta^S, \theta^u$ $\subset \theta^u$
 imply that g is PH

Containment $\Rightarrow$
 Existence $E^S, E^u, \dots$

Φ_{t_1} has $E^S, \dots$ has $\theta^S, \theta^u, \dots$
 $f = \Phi_{t_1} \circ D\alpha \circ \Phi_{t_0}$ of Φ_{t_1}
 E^S, E^S
of flow.

what we need is
 properties to prove $\left\{ \begin{array}{l} Dh^{-1}(E^S) \cap E^u \text{ (transversality)} \\ Dh(E^u) \cap E^S. \end{array} \right.$

this will imply cones for f



Because

$$- D\Phi_{\text{to}}(\mathcal{B}^u)$$

$$- Dh D\Phi_{\text{to}}(\mathcal{B}^u)$$

$$Dh(E^u) \cap E^s$$

$$- D\Phi_{\text{to}} \circ Dh \circ D\Phi_{\text{to}}(\mathcal{B}^u)$$

$$\text{conclusion } Df(\mathcal{B}^u) \subset \mathcal{B}^u.$$

same thing for other bundles $\Rightarrow$

f is PH.

$$D\alpha = h$$

changes topology

changes metric very little.

computation in hyperbolic plane

Also BGHP (Bonatti - Gogolev - Hammerlindl - Potrie) ^{S. Orientable}
(2017) generalize this to show for any mapping class β of S there is $f: T^1 S \rightarrow PH$

which induces β in $\text{Mod}(S)$.

In particular pseudo-Anosov mapping class.

Remark

PH can be obtained by the transversality conditions "as above".

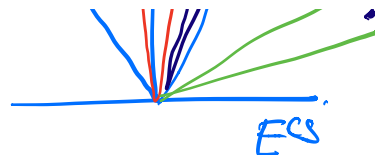
Also these ideas show that keep PH

is open in C^1 topology

$$E^s \cap E^u$$

$$\text{if } f \text{ is the PH } Df(E^s) = E^s \cap E^u$$

so g close to f satisfies this.



has comes properly g is also PH.

Conclusion Being PH is open. (in C^1 diffcos)

One goal - to understand PH in M^3 hyperbolic.

We obtain results in the opposite direction of BGHP.

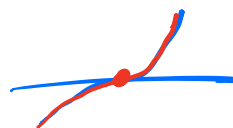
We need some definitions.

Def (TPAF) Topological Anosov flow - Φ flow in M^3 without stationary point orbits is topological Anosov flow if

- flow lines are C^1 stable unstable
- there are 2-dimensional foliations F^s, F^u flow saturated - so that
 - in a leaf of F^s all flow lines are forward asymptotic and diverge backwards, similarly for F^u in opposite direction
- the foliations are topologically transverse and locally intersect along flow segments.

- these properties are very close to saying flow is expansive.

topologically transverse



Ann

Def (DAF) Discretized Anosov flow.

(Let f be a PH) Then f is a DAF
if there exists Φ a topological Anosov flow
and $t(x)$ continuous in M so that
$$f(x) = \Phi_{t(x)}(x) \quad \text{for all } x.$$

(generalization of time one map of Anosov flow).
Some results.

THEOREM A (Barthelmé - Fenley - Frankel - Potrie BFFP)

Let $f: M \rightarrow M$ be PH and M^3 hyperbolic.
Suppose that f is dynamically coherent and
homotopic to the identity. Then f is
a discretized Anosov flow.

THEOREM B (Seifert manifolds) (BFFP) Let M be
Seifert fibered and $(\pi_1 M)$ not virtually solvable. Let
 f PH in M homotopic to the identity. Then
 f is DC and up to finite iterate it is DAF.

THEOREM C (F - Potrie) If M^3 hyperbolic
has a PH then M^3 has an
Anosov flow.

Why homotopic to the identity?

First recall that by Perelman -- (geometrization)
hyperbolic manifolds are by far the most

common amongst irreducible 3-manifolds
every sphere bundle $\supseteq$ ball.

If M^3 is closed hyperbolic then any f
homeo has an iterate which is homotopic to
the identity. This is because of Mostow rigidity
says that f is homotopic to isometry $\Rightarrow$
finite order.

- also if M is hyperbolic has a non trivial
isometry then one can possibly construct
examples - start with say PH finite order
map of Anosov flow on M hyperbolic. Lift
to finite cover and compose left of f
to a deck translate - this is PH not
homotopic to identity.

- This leads to statements^{in PH} that are up
to finite iterate and finite lift.

Branching foliations -

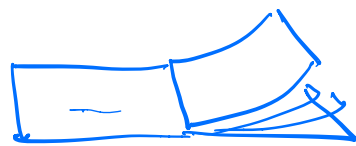
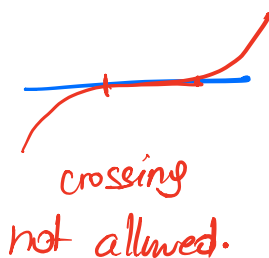
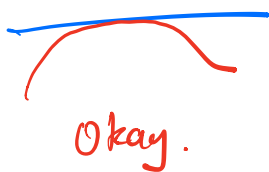
in general f PH on M^3 is not DC
(also in some BGHP examples - pseudo-Anosov)
So no foliations - But under certain
orientability conditions there are branching foliations.

This will be the main tool to study
topological theory of PH - at least in these

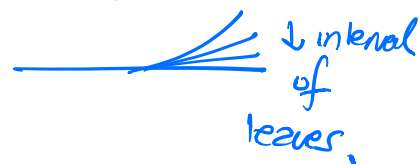
Def A branching foliation of a 3-manifold ^{results} is a collection $\mathcal{W}$ of C^1 -immersed surfaces each complete in its induced metric such that

- v) each $x \in M$ is contained in at least one leaf.
- vi) no leaf crosses itself.
- vii) different leaves do not cross each other.
- viii) if L_n are leaves of $\mathcal{W}$ and $x_n \in L_n$ converges to x then some subsequence of L_n converges to a leaf L with $x \in L$.

ix) - xii). crossing
no topological crossing



collapse here.
cross section



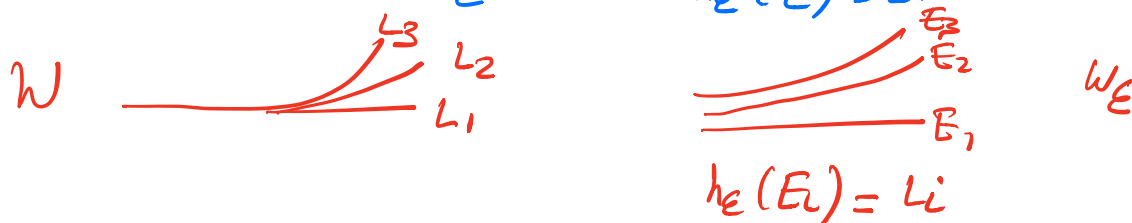
- ix) convergence understood in the compact open topology.

Additional good properties.

Def A branching foliation $\mathcal{W}$ is well approximated by foliations if there is a set of $\varepsilon > 0$ accumulating at 0 and a family of (actual) foliations $\mathcal{W}_\varepsilon$ with C^1 leaves and a family of continuous maps $\{h_\varepsilon: M \rightarrow M\}$ so that

(w. respect to fixed Riemannian metric)

- i) the angles between TW , Tw_ϵ are $< \epsilon$
- ii) the C^0 distance between h_ϵ and identity
- iii) on each leaf of w_ϵ the map h_ϵ restricts to a local diffeomorphism to a leaf of w
- iv) h_ϵ maps a leaf of w_ϵ to a leaf of w .
- v) For each leaf L of w there is a leaf L_ϵ of w_ϵ with $h_\epsilon(L_\epsilon) = L$.



Burago Ivanov. Jour. Modern Dynamics 2008

proved the following amazing result. no restriction

THEOREM (BI) Let f be a PH in $\mathbb{R}^3$ closed. no restriction

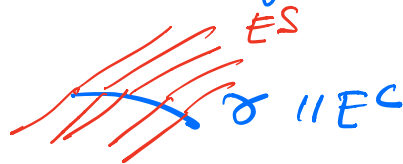
Suppose E^s, E^c, E^u are orientable and df preserves orientation. Then there are invariant branching foliations w^s, w^u tangent to E^s, E^u that are well approximated by foliations.

Ideas (very few)

recall that E^s is w^s uniquely integrable.

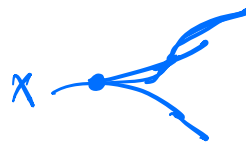
E^c is not uniquely integrable. But

There are integral curves.
 let γ be a small C^1 curve tangent to E^C
 1st result - Substrate by E^S
 $\downarrow$ local section is C^1
 and everywhere
 tangent to E^S .



- Not unique.
 of all local surfaces tangent to E^C
 through x take the lowest one there is one.

Now consider
 local such surfaces
 through every point



- there is an order -
 related to non crossing.



Then there is a very complicated iterated
 construction to put in more and more patches.

Obviously since start with lowest - this

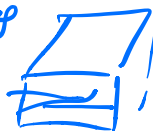
property is preserved by diffeomorphisms
 E^S and $\mathbb{A}$ orientable - E^C orientable
 if preserving it

BP Theorem 7.2 of
 Any branching foliation which is $\mathbb{A}$ orientable
 is well approximated by foliations.



used E^C orientable.

There is a procedure of local pushing
to separate.
use partitions of unity.



Some important results from 2-dim theory of 3-mfolds.
BI The approximating foliation does not have
Reeb components.

Novikov $\mathcal{F}^{2\text{-dim}}$ foliation in $\mathbb{R}^3$



Solid
torus

$\neq$ 2 leaf not π_1 -trivial

$\mathcal{F}$ orientable - if no

Reeb components (no solid Klein bottle Reeb components)

then - leaves of $\mathcal{F}$ are π_1 -injective

- closed transversals are not null homotopic.

Rosenberg $\mathcal{F}$ Reebless foliation
- shined M is irreducible.

M with PA is irreducible.

These facts imply that S^3 has no PH
 $S^2 \times S^1$ has no PH
finite $\pi_1(M)$ has no PH. has
Reeb
component.

For a while we study if PH homotopic
to the identity. $\pi_1(M)$ is not virtually solvable

In particular - branching foliations of BI
do not have compact leaves.

(HHU) - branching foliations - in $\mathbb{R}^3$ has

$T \subset$ leaves tangent to cu, su

Assume f preserves branching foliation as in BI.

We have W^{CS}, W^{cu} .

Def (good lift) Let f be homotopic to identity.

A good lift of f to $\tilde{M}$ denoted by $\tilde{f}$ is

one that

i) $\tilde{f}$ commutes with every deck transformation $\tau \in \pi_1(M)$.

ii) $\exists k > 0 \quad d(\tilde{f}(x), x) < k$ for all x in $\tilde{M}$.

This can be obtained for example by lifting a homotopy to identity to $\tilde{M}$.

In fact ii) follows from i) using fundamental domain.

Leaf spaces of $\tilde{W}^{CS}, \tilde{W}^{cu}$.

W^{CS} has no compact leaves

has no S^2, P^2 leaves because leaves of

W^{CS} are so saturated

+ Reebless $\Rightarrow$ leaves of $\tilde{W}_\varepsilon^{CS}$ are properly embedded planes.

by the maps $\tilde{h}_\varepsilon$ (good lift)

leaves of $\tilde{W}^{CS}$ are also properly embedded planes.

- Further result. Analysing proof of Thm 7.2 of BI - there are W_ε^{CS} so that

for each leaf L of $\mathcal{W}$ there is a unique leaf L_E of $\mathcal{W}_E^{cs}$ so that $h_E(L_E) = L$.

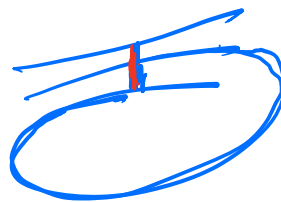
bijection between the leaves.

Leaf spaces in $\tilde{M}$ are homeomorphic.

Leaf spaces of Reebless foliations $\mathcal{F}$ without S^1 prs are one dimensional, ^{manifolds.} countable bases, possibly simply connected, possibly non Hausdorff

take a transversal τ to $\mathcal{F}$

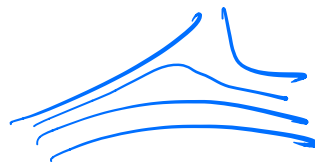
τ projects injectively to leaf space



if not perturb a path to a closed γ in $\tilde{M}$ - hull homotopic γ in M X .

$\Rightarrow$ leaf space $\mathcal{H}$ is 1-mfd.

simply connected - again ^{no} closed γ in $\tilde{M}$ but maybe non Hausdorff



Palmeira - showed $\mathcal{F} \simeq (\text{foliation of plane}) \times \mathbb{R}$ topologically