

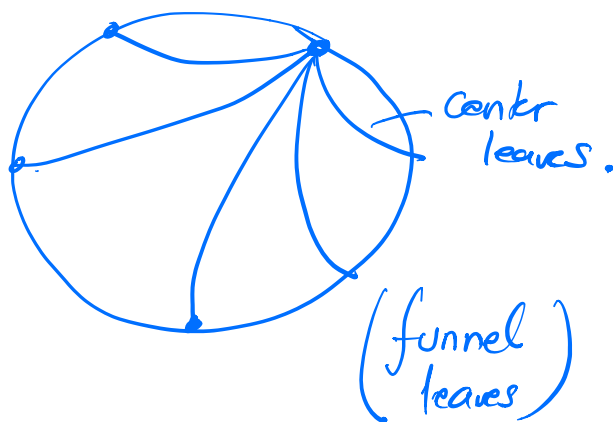
Mar 11. 2021.

One of goals is to show f PH in M^3

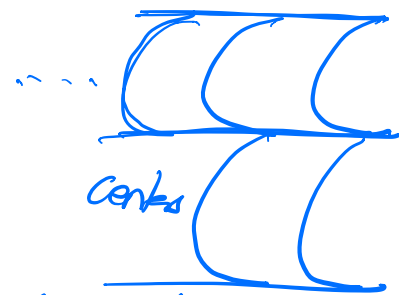
is in some cases DAF.

General comment - ^{one} possible structures of center foliation in leaves of $\tilde{W}^u$

for f DAF $L \in \tilde{W}^u$

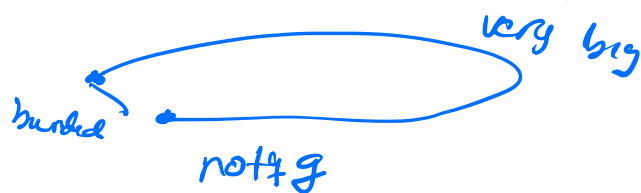


but not always
HHU example
leaves $L \in \tilde{W}^u$



In particular there is branching - leaves collapse.
Also center leaf space in L is
not Hausdorff.

Quasigeodesics uniformly efficient in measuring
distance length along curve is at most
 K distance between endpoints (for long enough distances)



Open question What are other possible structures
of center foliation in leaves of W^s (W^{cs})?
↓ also stable / unstable foliations?

Yesterday - we proved that if
 f PH homotopic to the identity, E^c, E^s, E^u
orientable Df preserves orientation. $\rightarrow$ we get
 W^cs, W^{cu} if they are f -minimal either
1) double invariance - every leaf W^s fixed
by f, W^{cu}

2) mixed

3) double translation

4) $\Rightarrow f$ is DC and f is DAF.

Today - bring back results to M

- translation in hyperbolic manifolds
- mixed case - no.
- DT case $\Rightarrow$ Answer flow in M .

$\Rightarrow M^3$ hyperbolic with PH then M^3 has

- 2 options
- DAF Answer flow.
 - DT (hypothetical).

Bring back results to M - we proved the
result for a finite cover + lift to finite
cover in order to get branching foliations.
(Burgio-Ivanov)

Proposition Let $f: M \rightarrow M$ be PH such that f^k is homotopic to the identity for some $k > 0$. Let $\tilde{M}$ be a finite cover of M which makes the bundles orientable. Let g be a lift to $\tilde{M}$ of a homotopy of f^k to identity so that g preserves orientation of the bundles.

Suppose g is dynamically coherent and that there is a good lift $\tilde{g}$ of g to $\tilde{M}$ that fixes all the center leaves. Then f is dynamically coherent and f^k is a discretized Anosov flow.

Application f PH is M hyperbolic.

$\exists k_0$ f^{k_0} hom. w. $\exists f^k$ power of f^{k_0} with finite lift g as above. If g is the double invariant case. then f is DC and f^k is a DAF.

Proof of the proposition

Invariance of branching foliations. Start with W^{cs}, W^{cu} for g . Then show that W^{cs}, W^{cu} are invariant under any deck transformation $\tilde{M} \rightarrow M$ (assume $\tilde{M} \rightarrow M$ is normal).
 lift to $\tilde{W}^{cs}, \tilde{W}^{cu}$ are invariant by $\pi(\tilde{P})$
 (i.i.)

↓ We show they are invariant by $T_1(M)$.

this implies that w^s, w^{cu} induce foliations in M (actual foliations). Then show $\tilde{w}^s, \tilde{w}^{cu}$ are invariant by any lift $\tilde{f}$ to $\tilde{M}$.

This implies that f preserves the induced foliations in M . $\Rightarrow$ f is DC.

Now $f_k \rightarrow g$ is DAF by previous arguments which implies that f_k is a DAF as well.

this uses a technical lemma.

g has a good lift
 f in general does not.

Lemma let g be

PH homotopic to identity w^s, w^{cu} g invariant foliations (not branching). w^s center foliation

Assume $\exists$ good lift $\tilde{g}$ fixes all leaves of $\tilde{w}^s$.

Suppose w_1^s, w_1^{cu} are also $\exists g$ -invariant foliations tangent E^s, E^{cu} . Suppose $\tilde{g}$ also fixes every leaf of $\tilde{w}_1^s$.

Then $w_1^s = w^s$ $w_1^{cu} = w^{cu}$.

Idea use w^{cu} and w_1^s, w^s .

2 pairs w^{cu}, w_1^s $\tilde{g}$ fixes all leaves of

W^{cu}, W^{cs} .

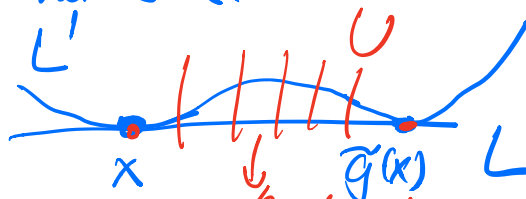
the lift,
 $\tilde{W}^{cs}, \tilde{W}_1^{cs}, \tilde{W}^{cu}$.

if $W^{cs} \neq W_1^{cs}$
 $\tilde{W}^{cs} \neq \tilde{W}_1^{cs}$.

there are 2 leaves
 $L \in \tilde{W}^{cs} \quad L_1 \in \tilde{W}_1^{cs}$

through same point but not same.

the U be
 center unstable of x .
 ↓ red.



iterate forwards —
 unstables increase contraction.

Mixed, double translation cases.

Seifert → one proof. DT

assume M hyperbolic.

goal → mixed does not happen
 → DT leads to know this in
 the manifold

First assume f M hyperbolic, $\hat{f}$ E^s, E^u, E^c
 one-to-one, Df preserving orientation
 $\tilde{f}$ good lift — $\tilde{f}$ acts as a translation
 on one of $\tilde{W}^{cs}$ or $\tilde{W}^{cu}$

Suppose wlog W^{cs} .

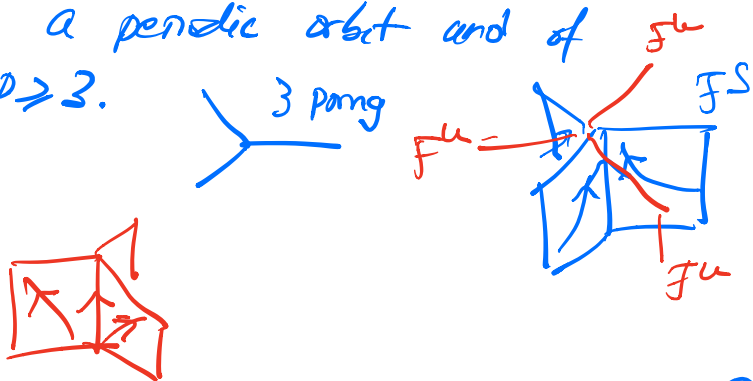
then W^{cs} is IR-covered, Uniform.

Proposition (BFP) for branching foliations.
 Under these conditions IR-covered in

M hyperbolic (need M 2-manifold) + M orientable then
 there is a pseudo-Anosov flow Φ transverse
 to W^{cs} and regulating for it.

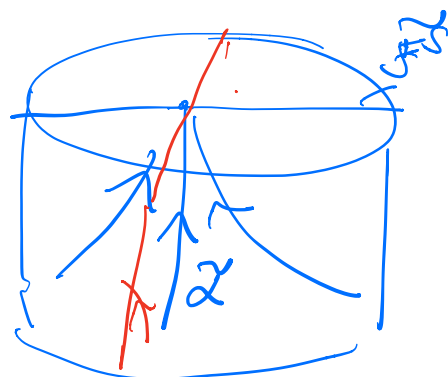
Pseudo-Anosov same conditions as TPAF

F^S, F^u (weak) stable/unstable but we
 allow singular orbits: at most finitely many,
 each such is a periodic orbit and of
 p -prong type $p \geq 3$.



Regulating means that every flow line of Φ
 intersects every leaf of $\widetilde{W}^{\text{cs}}$

Wants to get invariant
 sets associated with
 a periodic orbit α of Φ .
 Suppose $\gamma \in \pi_1(M) \leftrightarrow \alpha$



$\gamma(\tilde{\alpha}) = \tilde{\alpha}$ γ acts as translation in $\widetilde{M}$.
 I want to get an invariant leaf L of $\widetilde{W}^{\text{cs}}$
 by a power of $\tilde{f}$ so that we can
 analyse some action in L , - reducing the

analysis of F on 3-dim (F) to 2
 dimensions.

- However F acts freely. But combined with
 γ this is possible - we will obtain $h = \gamma^n F^m$
 $m > 0$ which fixes a leaf L of $\tilde{W}^s$ then
 we can understand dynamics of h on L .

- How do we get this fixed L ? branching
(below true for foliations)

Proposition M^3 hyp

$h: M \rightarrow M$ homeo homotopic to identity
 F branching foliation preserved by h , $\tilde{h}$
 good lift acts as a translation on $\tilde{F}$

let Φ be a regularizing, PD PA flow to F .

Then for every $\gamma \in \pi_1(M)$ associated with
 periodic orbit of Φ there is a compact
 (core) $\hat{F}_\gamma$ invariant set T_γ in

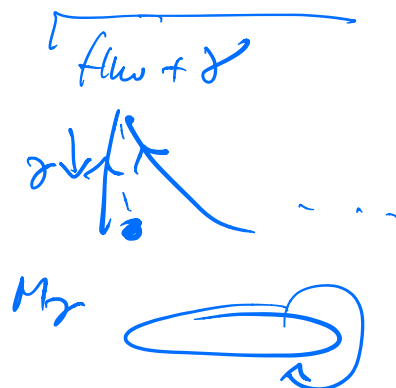
$$M_\gamma = M / \langle \gamma \rangle$$

$\hat{F}_\gamma$ induced by $\tilde{F}$ on

M_γ so that

T_γ intersects every
 leaf of $\tilde{F}_\gamma = \tilde{F} / \langle \gamma \rangle$.

and this consists
 of $x \in M_\gamma$ whose
 $\hat{F}_\gamma$ orbit

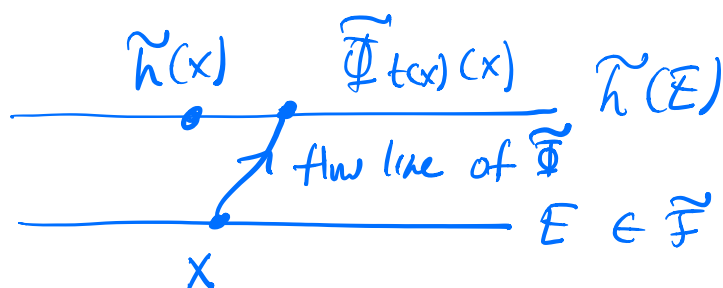


stays in a compact set in backwards and forwards time.

- Moreover if we denote $\tilde{f}_\gamma^k$ of $\tilde{f}_\gamma$ fixes a leaf L of $\tilde{f}_\gamma$ and γ fixes all the points of periodic orbit then the fixed set of $\tilde{f}_\gamma^k$ in L is contained in $T_\gamma \cap L$ and has negative Lefschetz index.

Apply to $\tilde{F} = \text{wcs } \tilde{h}(\langle \gamma \rangle)$.

Idea — use large scale properties.



Easy property — $d_{\tilde{h}(E)}(\tilde{h}(x), \tilde{\Phi}_{t(x)}(x)) < d_0$
is uniformly bounded

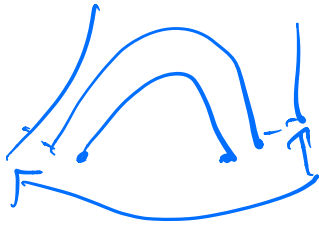
we know $d(x, \tilde{h}(x))$ is bounded by d_1

$$d(x, \tilde{\Phi}_{t(x)}(x)) < d_2$$

uses regularity + uniform. — uses compactness of M and limits

$$d(\tilde{h}(x), \tilde{\Phi}_{t(x)}(x)) < d_1 + d_2 \quad \text{in } \tilde{h}$$

$$\mathbb{R}\text{-covered} \Rightarrow d_{\tilde{h}(E)}(\tilde{h}(x), \tilde{\Phi}_{t(x)}(x)) < d_0$$



for $d_0 = d_0(d_1 + d_2)$.

distinct - this follows from local product structure of foliation

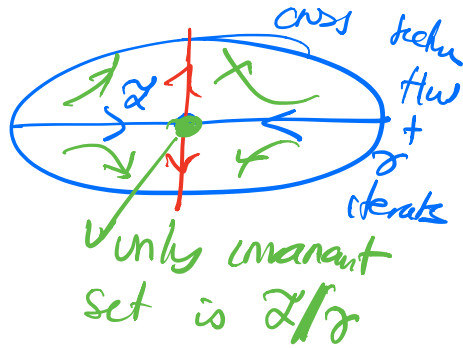
For flow in M_γ stable

using

$$d_{\tilde{H}(E)}(\tilde{H}(x), \tilde{\Phi}_{t(x)}(x))$$

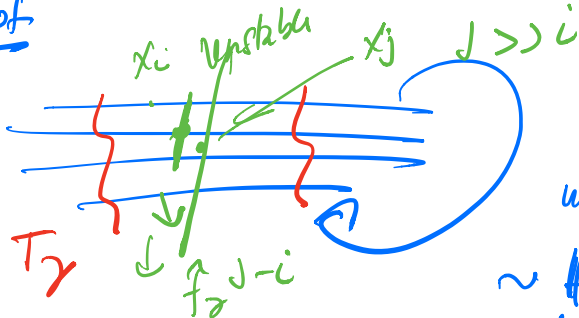
is bounded

implies the existence of compact set.



Lemma In our situation there is always a periodic leaf under $\tilde{f}_\gamma$ $\tilde{f}_\gamma^m(L_0) = L_0$

Proof



$$\tilde{f}_\gamma^m(L) = L$$

L a lift of L_0 .

$\tilde{f}_\gamma$. use green for unstable

in M_γ
 $\sim \mathbb{H}^2 \times S^1$
 $\sim$ leaves of $\mathcal{F}_\gamma$

- iterate x by $\tilde{f}_\gamma$ $\tilde{f}_\gamma^k(x) \in T_\gamma$

$$x_i = \tilde{f}_\gamma^i(x)$$

compact
 has an accumulate point.

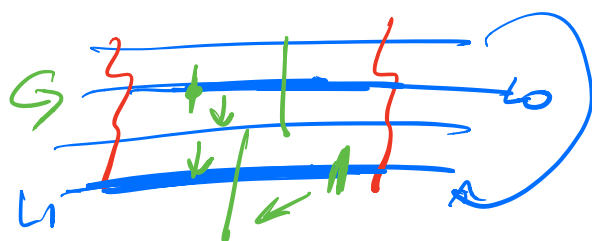
Locally f^{j-i} does the following interval of leaves

sent to bigger interval
 $\Rightarrow$ there is a fixed leaf
 by f^{j-i}



This lemma shows that under those ^{orientation} conditions M^3 hyp ... DT — f cannot be DC.
 (Theorem A under orientation condition).

if DC then foliation — also foliation in M_2

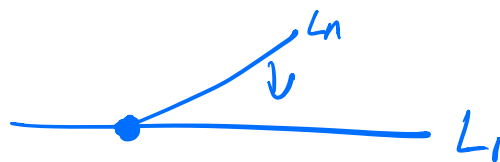


L_0 fixed by f_k .
 M_2 f_k and expands away from L_0

if it expands somewhere it contracts somewhere else

But contraction is impossible by same argument applied to L_1

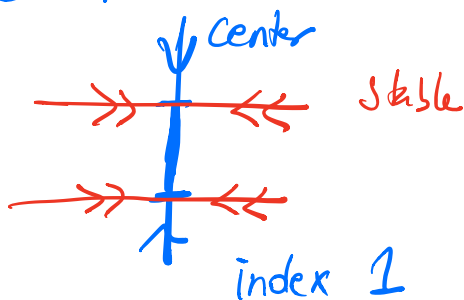
Be aware this uses DC foliations.



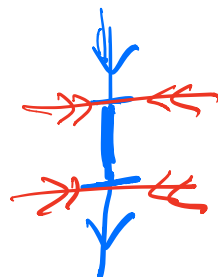
branching
 makes it much
 harder to
 analyze.

here branching case is substantially different.
Lefschetz index, no general definition.

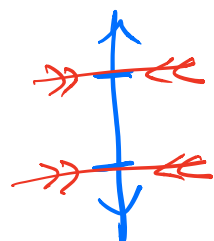
examples, - in $\tilde{W}^{\text{es}}$



coarsely contracting center
 red = stable
 blue = center

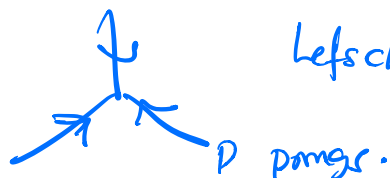


index 0
 fixed point could be removed



index -1
 expansion on stable.

pseudo hyperbolic orbit - p prongs.



Lefschetz index = $1 - p$

Mixed case - L fixed leaf of $h = \gamma f^m$

$m > 0$ γ associated with $\geq p$ prong orbit

$p \geq 3$.

This always exists by Poincaré

Lefschetz index is negative $1 - p$.

This leads to an index computation -

we find a finite number of fixed centers

by h in L and these (with $p \geq 3$)
lead to a contradiction - technical.

Lefschetz theorem - Lefschetz index $\neq 0 \Rightarrow$
fixed points - fixed point leads to fixed
center.

Property Fixed center by h in L can only
accumulate in ideal point of primer of $\mathcal{Z} \cap L$
 $\alpha \leftrightarrow \gamma$.

$$X = \mathcal{Z} \cap L$$

assume regular orbit

$$h(C) = C$$

using dynamics of h of flow
controlled in large skeletons

fixed points in compact part

redraw with $h(C) = C$

C center

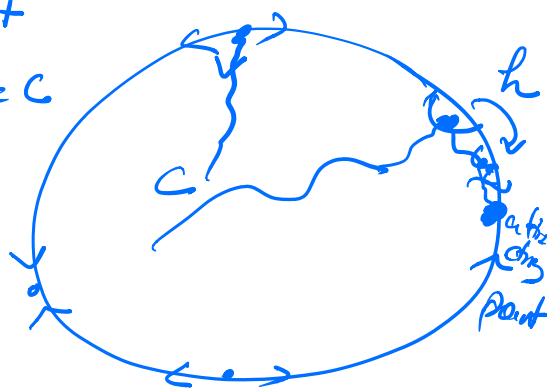
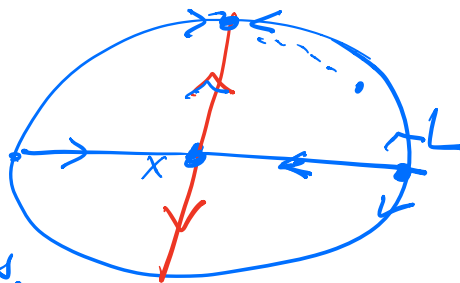
suppose C accumulates

on point not

attracting repelling

$\downarrow$ then C is forced to accumulate only
on an attracting point in $S^1(L)$.

show finitely many such C - index computation
leading to a contradiction of $p \geq 3$.



If no such α then Φ is TPAF.

this cannot happen for regulating TPAF in

$\mathbb{R}$ -covered foliation -

This rules out the mixed case.

Finally divide translation case. (F-Potrie)

We are assuming M hyp f PH homotopic to id .

W^s, W^u E^s, E^c, E^u unoriented of preimage unoriented

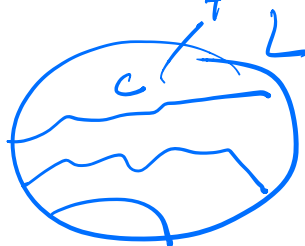
f good lift, f acts as a translation on W^s, W^u .

Strategy involves much more geometry

2 main steps.

1) quasigeodesic behavior.

Show that for any leaf $L \in \tilde{W}^s$
any center leaf c in L is quasigeodesic
a uniform quasigeodesic.

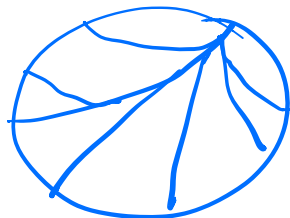


2) show this implies
center foliation is a collapse

of the flow foliation of
a topological Anosov flow

$\Rightarrow$ there is TPAF
in M

M hyperbolic



$\Rightarrow$ (Marker) TPAF is transitive

$\Rightarrow$ (Shenau, 2020) TPAF is orbitally
equivalent to an
Anosov flow (C')
with bundles etc..

This is first in finite cover - then
project to M using invariance
of foliations.

How to get 1) quasigeodesic behavior?

This is done in many steps.

I) For any $L \in \widetilde{W}^s$ any ray r in
either a stable leaf or center leaf in L
then r converges or accumulates in a single
point in $S^1(L)$.

Suppose not
do it for center
foliation



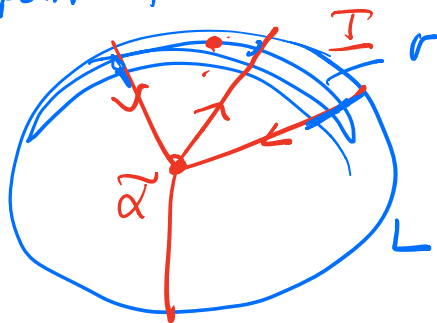
Some ray r in some L
accumulates to a non trivial interval I in $S^1(L)$

Zoom in to an interior point of I

Now choose $\gamma \in \Pi_1(M)$
with $\tilde{\gamma}$ associated orbit
periodic

so that at least

3 prongs of $\tilde{\gamma}$



intersect this ray only many times.

Uses compactness of M and density of periodic orbits of Φ .

Apply $h = \gamma^n \gamma^m$

so that it fixes E^1

$m > 0$



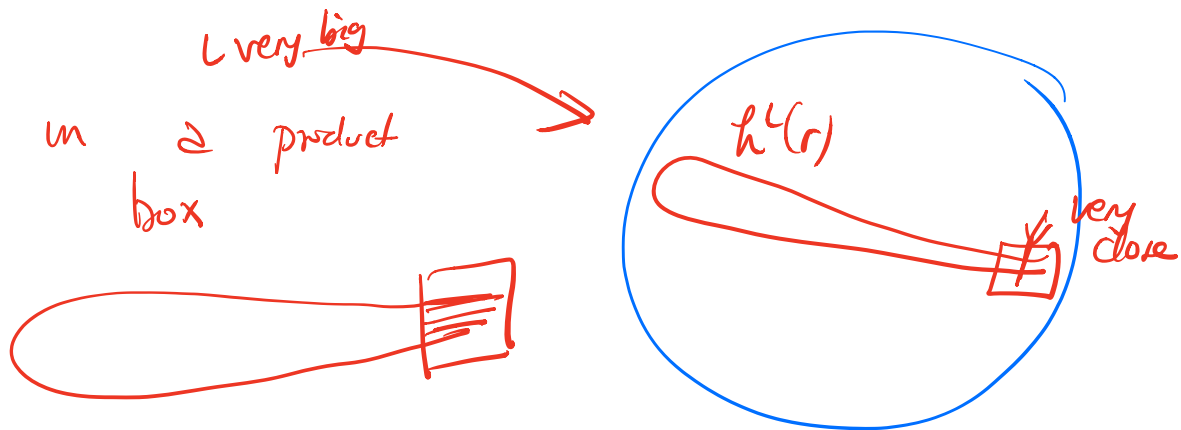
$h^i(L) \rightarrow E_0$ fixed by h . $h^2 E_0$

$$h \gamma(E_0) = \gamma h(E_0) = \gamma(E_0)$$

$h = \gamma^n \gamma^m$ γ commutes with γ

$h^i(r)$ are in leaves $h^i(L) \rightarrow E_0$

contraction of some points towards a core



Impossible by index computation

This is the argument to show that rays converge to single points.

Two more properties

small visual measure of arcs in stable unstable

↓

given $d > 0$ $\exists \theta_d > 0$ visual angle

so that if γ is a segment in stable
or center leaf in L and $d(x, y) > d$
for all $y \in \gamma$

then visual measure in L
from x of γ
is $< \theta_d$

$$\theta < \theta_d$$

Small visual measure $\theta_d \rightarrow 0$
as $d \rightarrow \infty$.

if not

This implies another
property:

$\exists$ global $d_0 > 0$
if γ is a ray or compact
segment in a leaf L of $\tilde{W}^s$

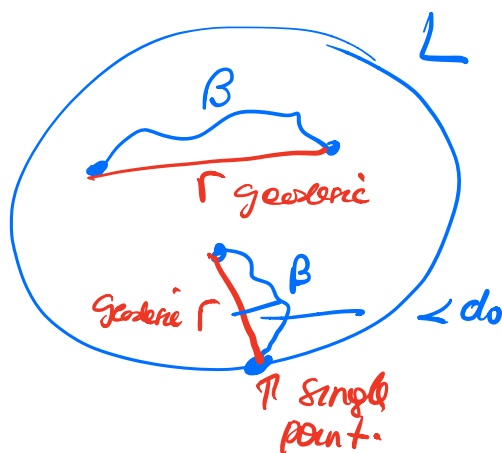
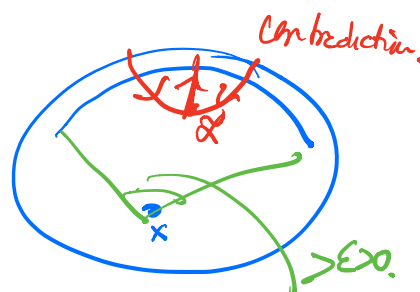
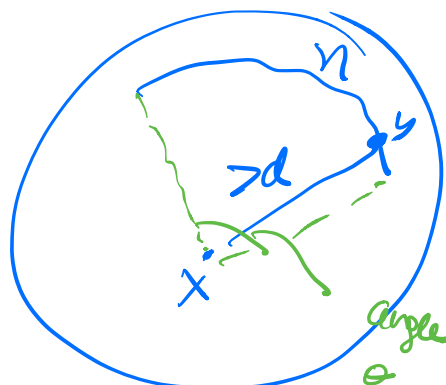
and r is the geodesic arc or ray
with same endpoints

Property

$$r \subset N_{d_0}(B)$$

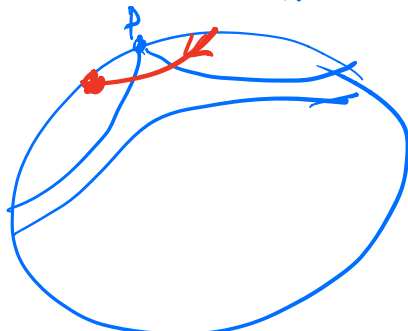
$\downarrow$
neighborhood of size
 d_0 in L

Watch out converse not necessarily
true.



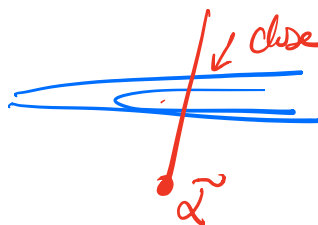
Proposition $L \in \tilde{N}^{\text{reg}}$ the center foliation
in L has Hausdorff leaf space

Idea if not $\exists \tilde{\gamma}$
if $\exists \tilde{\gamma}$



then iterate by $h = \gamma^n \gamma^m \rightarrow$

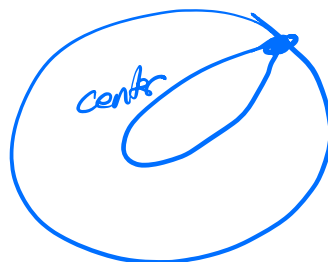
contradiction
leaves intersecting foliated
box more than once.



Not always possible because maybe p is
ideal point of a stable lamination so we
have no such intersection.

Because of that we need to analyze
centers with both ideal points equal.

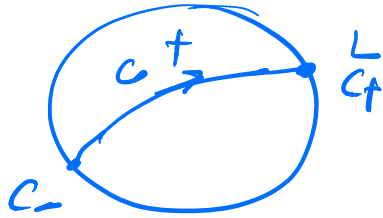
this happens for
stable foliation.
(eg DAF).



this
does
not
happen

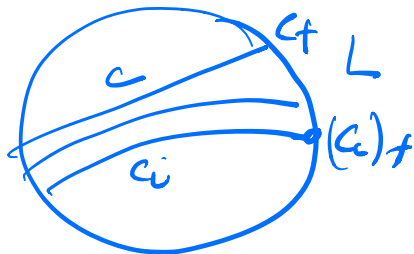
Proving this is technical
and a lot of work.

orient E^c in $\tilde{M}$



c_+ ideal point of
ray in positive
direction - similarly
for c_-

Lemma c_+, c_- vary continuously with c



$$(c_i)_+ \rightarrow c_+ \text{ in } L$$

but also the c_i are in distinct leaves
use topology in $US^1(L)$
 $L \in \overline{Wes}$

using visual circle at each point

$$S^1(L) \sim T^1 \tilde{F}|_{K_L}$$

$$\tau \cap \tilde{F}$$

maps $US^1(L)$ into annulus,

$$L \cap \tau \neq \emptyset$$



Result c_+, c_- exist

$$c_+ \neq c_- \neq c$$

c_+, c_- continuous

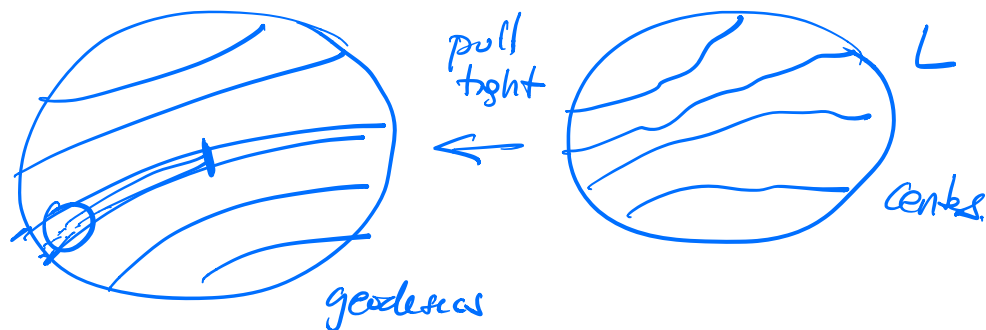
implies that the c foliation in any L

is by uniform quasigeodesics,

this does Step 1 $\rightarrow$ center

Step 2 $\rightarrow$ TPAF.

- in each leaf you can pull tight geodesics



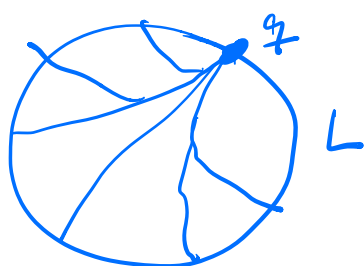
by zooming in technique (Calegari)

$\Rightarrow$ there is a leaf where the geodesic foliation is $\geq$ funnel



$\mathbb{R}$ -covered $\Rightarrow$ this happens everywhere.

For original center foliation every leaf



all centers share
an ideal point.
funnel leaf $L \in \mathcal{L}_0$

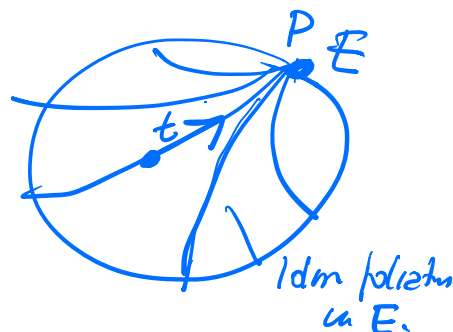
to get Anosov flow - consider a foliation

W^s_E - this flow can be put in a metric
w.r.t so that all leaves
are hyperbolic

for each leaf E of $\tilde{W}_E^{CS}$ consider the
 ideal point in $S^1(E)$ corresponding to funnel
 in $\tilde{h}_E(E) = L$

$$p \leftrightarrow q$$

in E consider the
 foliation by geodesics
 with ideal point p .



This varies continuously with $E \rightarrow$ uses
 $\tilde{W}_E^{CS}$ is $\mathbb{R}$ -covered centres
 are uniform $\mathcal{G}$

at this point we have a 1-dim foliation in $\tilde{M}$
 in each leaf E of $\tilde{W}_E^{CS}$ it is this
 funnel foliation by geodesics.

This projects to a 1-dim foliation in M
 take flow that is future distance t in
 the leaf in the direction of funnel.

Flow in M - show this flow is
 expansive -

Expansive any 2 distinct orbits diverge from
 each other in future or backwards time
 clear in leaf of $\tilde{W}_E^{CS}$. Also prove
 transversely the opposite happens.

$\downarrow$ Paterchia $\Rightarrow$ flow is pseudo-Anosov
 $\downarrow$ M but preserves an actual foliation
 $\downarrow$ Matsumoto $\nu_{\Sigma}^{CS} \Rightarrow$ flow is
 TPAF - ^{topological} Anosov flow.

Done in case of orientable conditions.

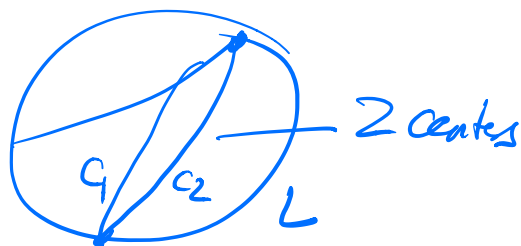
Remark - The center foliation can be obtained
 as a collapse of the flow foliation
 (Collapsed-Anosov flows
 Barthelmé - F - Potrie)

- technical issue that we did not cover
 funnel condition

~~can~~ has to show
 any 2 distinct

centers c_1, c_2 in L

cannot share both ideal points.



THANK You !!