

Mar. 8, 2021

PARTIALLY HYPERBOLIC DIFFEOS

in DIMENSION 3.

Dynamical systems - study long term behaviour
of $f: X \rightarrow X$ homeomorphism (usually X compact manifold, or X')
or flows $\phi = \phi_t$ $t \in \mathbb{R}$.

Several aspects

- 1) iterates of $f^n(x)$ and where they accumulate / converge.
- 2) are there periodic orbits? How many?
What is the growth of number of periodic orbits with period? Entropy?
- 3) Are there dense orbits? Is the diffeo transitive?
 $\Rightarrow$ union of periodic orbits is dense
(periodic orbits say a lot about the system).
- 4) study of invariant measures - ergodic theory.

Def (partially hyperbolic diffeomorphism) ^{70's - Hirsch-Pugh-Shub}

$f: M^n \rightarrow M^n$ M Riemannian manifold, $f \in C^1$
diffeo is partially hyperbolic = PH if there is
 $\exists$ Whitney decomposition

$$TM = E^s \oplus E^c \oplus E^u$$

{ continuous
bundles.

E^s stable, E^u unstable, E^c center

which are Df invariant $\forall x \in M$

$$\forall v^s \in E^s, v^u \in E^u, v^c \in E^c \quad \|v^s\| = 1$$

$$\|Df^i(v^s)\| < \max\{1, \|Df^i(v^c)\|\} \leq \max\{1, \|Df^i(v^u)\|\}$$

$v^s \in E^s(x) \quad v^u \in E^u(x) \quad v^c \in E^c(x)$
 $Df^i|_{E^s(x)}$ is contracting.
 $Df^i|_{E^u(x)}$ is expanding
 by compactness $Df^i|_{E^s}$ is contracting.

E^s is a contracting subbundle
 E^u is an expanding subbundle

this is sometimes called pointwise PH.

- There is also absolute PH $\exists \lambda, \beta$
 $\lambda < 1 \quad \beta > 1$

$$\|Df^i(v^s)\| < \lambda < \|Df^i(v^c)\| < \beta < \|Df^i(v^u)\|$$

no matter which point $x \quad v^s \in E^s(x)$.
 unit vectors

- A lot of times absolute is stronger and produces stronger results.

Remark using averaging, adapted metric one can assume that $\lambda = 1$.
 we assume that.

E^s, E^u are what is called hyperbolic bundles
 - E^s contracts, E^u expands.

Theory of hyperbolicity (integrability of bundles)

Anosov 60's, Hirsch-Pugh Shub 70's. If E^S is a stable bundle, E^u unstable bundle for a diffeomorphism (or flow) then they are uniquely integrable.
(no need of PH or both bundles)

Integrable means that there is a foliation

$$f^S \quad \dim F^S = \dim E^S \quad T F^S = E^S \quad \text{leaves of } F^S$$

and $f(F^S) = F^S$ f invariant



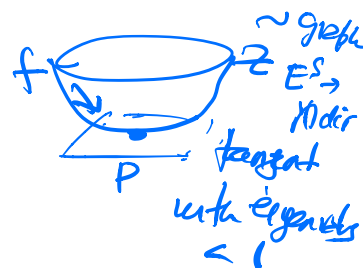
- This transforms information about the tangent bundle to geometric information in M .

Why? simplest case is p is hyperbolic fixed point. $f(p) = p$ $Df(p)$ has eigenvalues all < 1 or > 1 in size.

(local) Stable manifold theorem If $f: M \rightarrow M$ diffeo
 p hyperbolic fixed point then $\exists$ local submanifold Z of M $p \in Z$ (stable nbd of p)
 $\dim Z = \text{number eigenvalues } < 1$, $f(Z) \subset Z$

$$\forall w \in Z \quad f^n(w) \rightarrow p \quad n \rightarrow \infty$$

and if y is near enough p
and $f^n(y)$ in nbd of p all $n > 0$
then $y \in Z$.



This is a classical result using linear algebra and analysis in Banach spaces. Banach spaces of graphs of Lipschitz functions form subspaces of $T_p M$ - use a graph transform - show this is a contraction and has a unique fixed point. not true!

why study PH? Because they are essentially the ones which have robust properties. For example

Theorem (Diaz - Ruels-Ures) If g in $\text{den } 3$ is robustly transitive then either f is Anosov (PH without EC), or PH or what is called weakly PH $\Rightarrow$ has a dominated splitting. (two bundles are stronger than other)
robustly transitive - transitive in a small $\mathcal{C}^1$ nbhd of g . $\rightarrow$ one main focus of dynamical systems.

$\downarrow$ Anosov - has $TM = E^s \oplus E^u$ with all proper of PH.

by Anosov - if f Anosov and transitive then it is robustly transitive.

f is structurally stable - nearby h is equivalent to f

For a long time Anosov were the only examples of robustly transitive. Then in 90's examples of robustly transitive + none Anosov were

constructed - these were all PH.

Theorem (Boratti-Diaz) Smaller result if f is stably ergodic. Means ergodic w.r.t. cl^d nbhd of volume preserving diffeos. (probability measure)
 f ergodic $f(A) = A \Rightarrow m(A) = 1$ or $m(A) = 0$.

Most of the course we will be in dimension 3. Main examples in dimension 3.

i) (perturbations of) f linear Anosov diffeo of T^3 into 3 distinct eigenvalues $\lambda_1 < \lambda_2 < \lambda_3$

all real say > 0 .

$\lambda_i \rightarrow$ eigenvector v_i induces foliation in T^3 in direction v_i

the bundles tangent to these are of invariant -

$\lambda_1 < 1$ $\lambda_3 > 1$ $v_1 \rightarrow E^s$ contracting
 $v_3 \rightarrow E^u$ expanding

$\lambda_2 \rightarrow v_2$ is in between $v_2 \rightarrow E^c$ is between

in particular if $\lambda_2 > 1$ E^c is expanding everywhere
if $\lambda_2 < 1$ E^c is contracting
if $\lambda_2 = 1$ E^c is invariant of size

(generalize to higher dimension

T^n .

$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$
 < 1 strictly between other ones. > 1

0.2, 0.3

0.7, 0.7, 1.2, 1.2

2, 5

2) (perturbations of) skew products.

 $M = T^3$ or
nil/manifoldssimplest $M = T^3 = T^2 \times S^1$ start with $g: T^2 \rightarrow \text{Anosov}$ say linear with eigenvalues $0 < \lambda_1 < 1 < \lambda_2$. $f: T^2 \times S^1 \rightarrow$ $f(p, t) = (g(p), h_p(t))$

where

 $\lambda_1 < \|Dh_p(v)\| < \lambda_2$ E^S $\rightarrow \lambda_1$ $\neq 0$

unit vector in vertical direction

 $E^u \rightarrow \lambda_2$ E^C $\leftrightarrow$ tangent
to $p \times S^1$ here at different points we may have
contraction or expansion in E^C .

- clearly you can generalize to higher dimensions.

 $T^n \times T^2$ Anosov in T^n in T^2 not as much expansion /
contraction.

3) (perturbations of)

time one maps of Anosov flows.

Def Anosov flow in M^n Φ in M^n smooth (C¹) $TM = E_{\Phi}^S \oplus T\Phi \oplus E_{\Phi}^u$ $\downarrow$
stable $\downarrow$
unstable $\downarrow$
 $\exists c > 0, \lambda < 1$

invariant by

bundle

 $D\Phi_t$ $\forall t$ given $v \in E^S$

similar for E^u $\|D\Phi_t(v)\| \leq c \lambda^t \|v\| \quad t \geq 0$

Example Φ geodesic flow on $M = T^1 N$
 where N is a closed manifold with negative
 sectional curvature (Anosov, Hopf)

for surface N
 in $\tilde{N}$

tangent to horocycles
 gives E^S .



the PH associated to it

$$f = \Phi_1$$

$$E^S = E_{\Phi}^S \quad E^u = E_{\Phi}^u \quad E^c = T\Phi,$$

one can also do variable time (smooth c')

$$f(x) = \Phi_{t(x)}(x) \quad t(x) \in c'$$

this is PH $E^c = T\Phi$ but E^S, E^u not
 necessarily E_{Φ}^S, E_{Φ}^u .

Integrability

Φ Anosov

hyperbolic bundles



also works for flows.

But Anosov proved that

— go back to Anosov flow

$$TM = E_{\Phi}^S \oplus T\Phi + E_{\Phi}^u$$

E_{Φ}^S, E_{Φ}^u integrate uniquely
 to foliations.

$$\begin{matrix} E^S \oplus T\mathbb{R} \\ E^u \oplus T\mathbb{R} \end{matrix}$$
 uniquely integrable to foliations as well.

In dim 3 $\Rightarrow$ a pair of 2 dimensional foliations transverse to each other.

Go back to f DH $E^S \oplus E^C \oplus E^u$

define $E^S = E^C \oplus E^S$ center stable

$E^u = E^C \oplus E^u$

2 questions
 I want f invariant foliations.
 A) are E^S, E^u integrable to foliations?
 B) is E^C integrable to a foliation?
 Uniquely?

In general the answer is NO.

There are simple counter examples in higher dimension in nilpotent manifolds - Wilkinson

The non integrability in those cases has to do with the failure of the Frobenius condition for integrability of bundles.

From now ON unless otherwise stated we restrict to dimension 3.

bundles E^S, E^C, E^u are one dimensional.
 integrability is easier to understand.

locally E^S ... are just vector fields
 If E^C is C^1 then uniquely integrable.
 If E^C is Lipschitz then uniquely integrable.
 But in general the bundles are only Hölder.
 There are rigidity results if one assumes a lot
 of differentiability (Angebot für Schatzkammer).

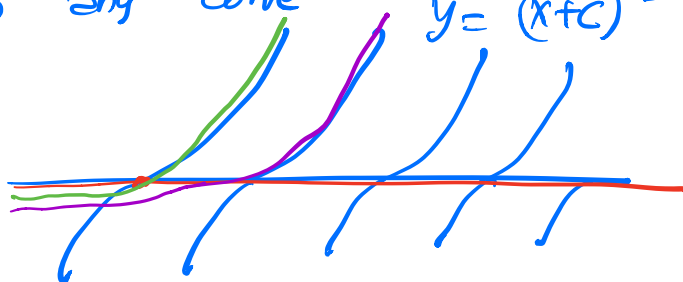
Example in $\mathbb{R}^2$ - x, y plane - consider
 differential equation

$$\frac{dy}{dx} = 3y^{2/3}$$

this is Hölder but not Lipschitz

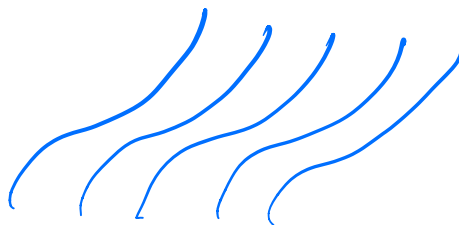
x axis is a solution $y \equiv 0$

but so is any curve $y = (x+c)^3$ c is a constant

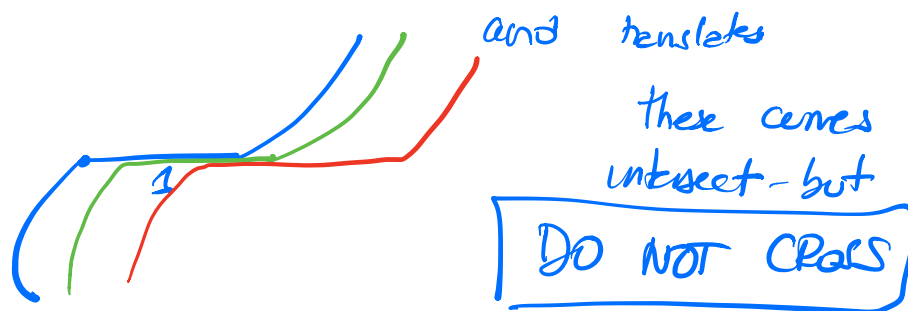


non uniqueness of solutions (even locally) through
 x axis

you can produce foliations



but also "branching" foliations - more than one curve
 through a point



Def f PH (any dimension) is called dynamically coherent (DC) if there are foliations $\mathcal{F}^cs, \mathcal{F}^cu$ f -invariant $T\mathcal{F}^cs = E^cs, T\mathcal{F}^cu = E^cu$. In particular $\mathcal{F}^cs \cap \mathcal{F}^cu$ is a foliation f -invariant and integrates E^c .

Question Are all f PH in dim 3 DC?
in dim 3 $\dim E^c = 1$ through any point there is an integral curve

answer is No because E^c is continuous.

THEOREM (HHU - Rodriguez-Hertz, RodriguezHertz, UAS) took a long time - 2014

There is f PH in T^3 homotopic to skew product and so that f is not DC.

we explain - start with (A) linear Anosov diffeo of T^2 with eigenvalues $\lambda, 1/\lambda$ $0 < \lambda < 1$

let e_s be vector field in T^2 // stable direction

2 steps 1) $f_0: T^2 \times S^1 \rightarrow$

$$f(p, \theta) = (A_p, \tau(\theta))$$

$\downarrow$
 NS map



$\rightarrow$ vs $\Rightarrow$
 attracting $\downarrow$ more attracting.

τ has properties below $\tau'(0) = \mu > 1$
 $\tau'(1/2) = \sigma < 1$

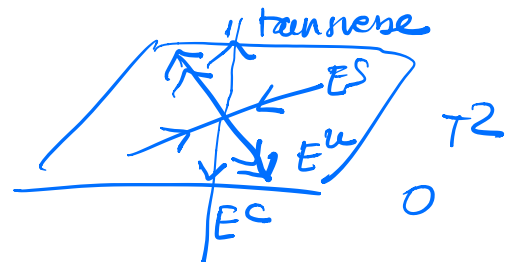
$$0 < \sigma < \lambda < 1 < \mu < 1/\lambda$$

contraction more
 than expansion

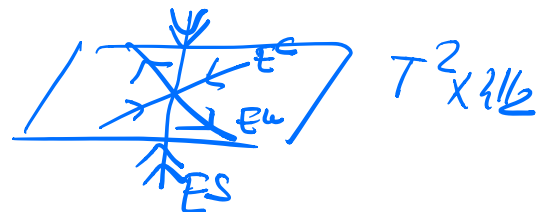
expansion
 less than contraction

picture

expectation



EU same
 ES, EC ?

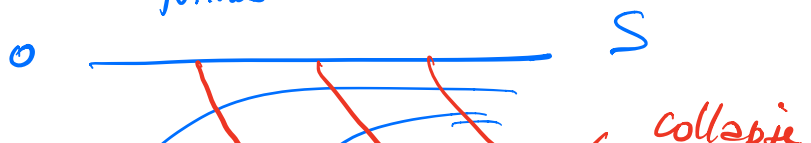


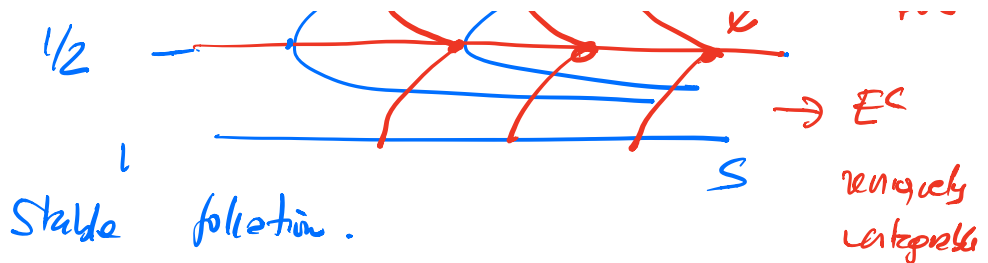
$$\begin{aligned}
 &2) f(p, t) \\
 &= (A_p + \underline{v(\theta)} e_s, \tau(\theta))
 \end{aligned}$$

$$v(0) = 0 = v(1/2) = 0$$

EU the same ES, EC change depending on θ .

on the S_1 vertical direction the vector fields
 integrate to as follows





Stable foliation.

$S = \text{stable}$

$C = \text{center}$

- one can do this (HVV)

so that f is PH.

But it is not DC, - collapse of center curves.

Center curve - a curve tangent to EC.

Two properties - this example has

{ a torus tangent to CU $\theta = 1/2$.
also a torus tangent to SU $\theta = 0$.

these are extremely rare - but very bad for PH.

notice this example is not transitive

there is a torus that is an attractor $\theta = 1/2$

general reference for PH

article $\rightarrow$ Crowder-Potrie. $\rightarrow$ in Rafael Potrie's webpage.
notes