

Mar 10 -

Set up - f PH on M^3 closed.

$\pi_1(M)$ not virtually solvable
 f homotopic to the identity.

Applications to M hyperbolic, Seifert.

Up to deck + finite cover we have the
orientability conditions - Burago-Ivanov

✓ $\exists$ f invariant branching foliations w^s, w^u

So in general we take finite deck and lift g
and prove smtg for g . Then we have to
project down to M and get information for f
if possible. This is associated with invariance
of foliations under deck transformations and f .

So we assume we took this deck + finite lift.
Our big goal is to show that under certain
circumstances f is DAF - discretized Anosov

that is there is Φ topological Anosov flow
and $f = \Phi_{t(x)}(x)$

General idea of what we will do.

Suppose $f = \Phi_{t(x)}(x)$ for some TPAF Φ

Φ has (weak) stable, unstable foliations

FS, FU . let $\tilde{f}$ the lift = $\tilde{\Phi}_{t(x)}(x)$.

then for any $L \in \tilde{F}^S$, $U \in \tilde{F}^u$ ^{Good lift.}

$$\tilde{f}(L) = L$$

$$\tilde{f}(U) = U$$

in addition if

$$\beta \text{ is } \beta = L \cap U \text{ for some } L \in \tilde{F}^S, U \in \tilde{F}^u$$

a single component, an orbit of $\tilde{\Phi}$.

$$\text{again } \tilde{f}(\beta) = \beta.$$

in this case $\tilde{f}$ preserves every leaf $\tilde{F}^S, \tilde{F}^u, \tilde{\Phi}$.



This is what we want to prove in certain cases:

$$\begin{array}{ccc} \tilde{F}^S & \longleftrightarrow & \tilde{W}^{CS} \\ \tilde{F}^u & \longleftrightarrow & \tilde{W}^{CU} \\ \tilde{\Phi} & \xrightarrow{??} & \boxed{\text{Center foliation}} \end{array}$$

branching 1-dim foliation

Center foliation

A center leaf in $\tilde{M}$ is a connected component of the intersection $E \cap V$

E leaf of $\tilde{W}^S$, V leaf of $\tilde{W}^u$

we do not know they intersect in a single component.

we try to show that $\tilde{f}$ fixes leaves of these $\tilde{W}^S, \tilde{W}^u, \tilde{W}^c$

$\tilde{W}^c$ center foliation in $\tilde{M}$, W^c projection to M .

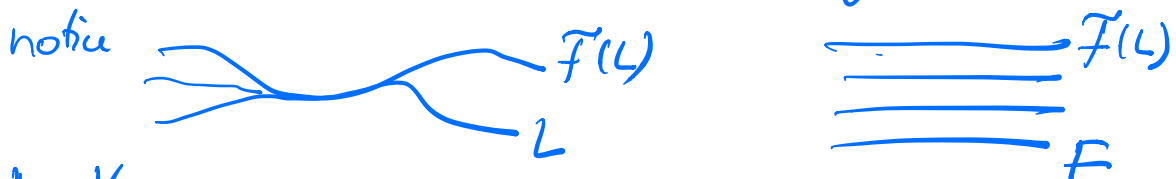
understand this.

General remark. Most properties and proofs work equally well for branching and unbranching foliations (in case W^S is an actual foliation). But sometimes the proofs are harder or much harder. Sometimes the results do not work in branching case.

General strategy - build up structure little by little.

Proposition Let L be a leaf of W^S (or W^{cs})

Suppose that $\tilde{f}(L) \neq L$. Then $L, \tilde{f}(L)$ bound a closed region in $\tilde{M}$ transitively foliated - the leaf space restricted to this region is $[0, 1]$.



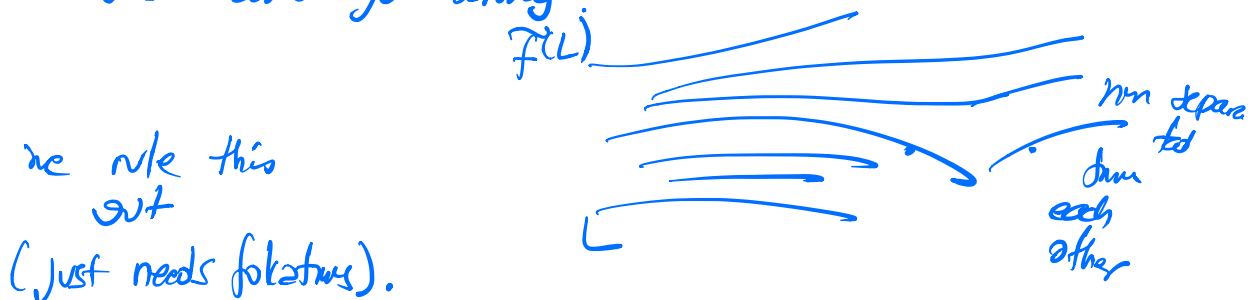
possible

Proof (this result really is for foliations - uses f homotopic to id W_E^S Reekless, no compact leaves)

(reference BFP - Denker-Helmke - F - Potrie

for 1-1 $h \in W_E^{cs} \rightarrow W^{cs}$).

What could go wrong?



Property. For any leaf L of $\tilde{W}^s$ choose a
 let $L_+ = L \cup$ positive compl. component
 to $\tilde{W}^s$

L_-

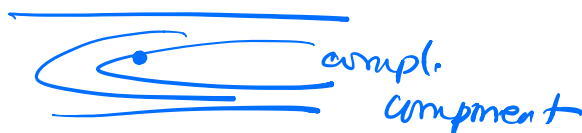
Property. for any L, L_+, L_- contains
 $B_r(x), B_r(y)$ ($r > 0$)
 $(F \sim 2000)$
 $C \subset L_+$ $C \subset L_-$

↓ this uses that W^s, W^{cu} are "tut".
 there is $\geq$ closed transversal through every leaf.
 stronger than Reebless

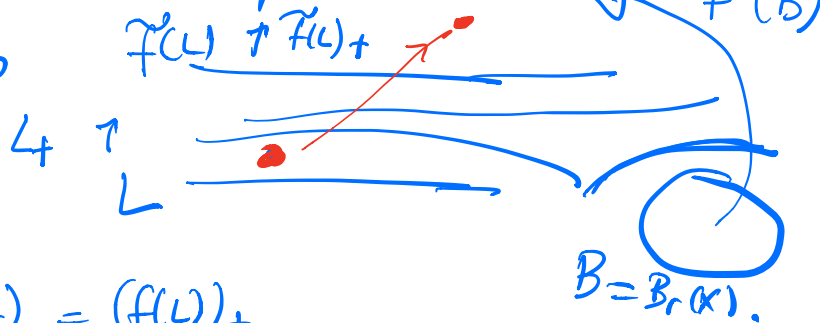


Reebless, not tut
 $T^2 \times I$

$T^2 \times I$



Now back to

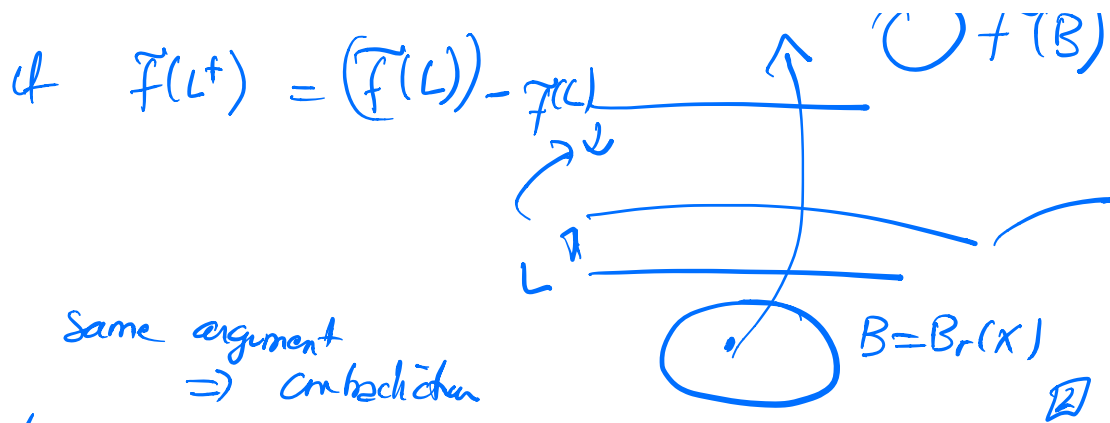


Suppose $f(L_+) = (f(L))_+$

in particular $f(B) \cap B = \emptyset$

but recall $d(f(x), x) < \epsilon$ for all x

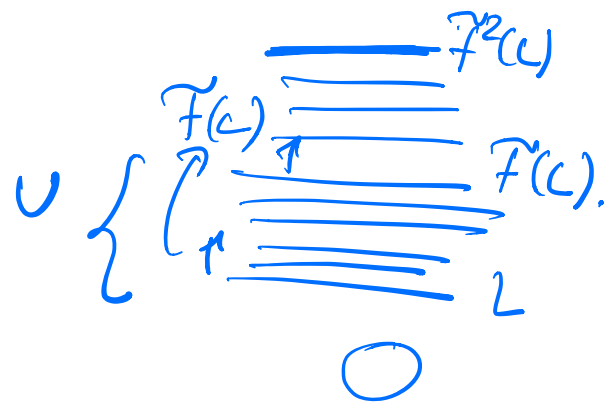
impossible if $r > \epsilon$.



What was used was $\cdot d(F(x), x) < \epsilon$ } foliations
 + that no PH needed.

More structure

Let $U =$ region
 between $L, F(L)$
 including endpoints



Let

$$V = \bigcup_{n \in \mathbb{Z}} F^n(U).$$

in V the foliation $\tilde{W}^cs$ restricts
 to a foliation with leaf space $= \mathbb{R}$.

Proposition. Suppose $L \in \tilde{W}^cs$ $F(L) \neq L$.

$$V = \bigcup_{n \in \mathbb{Z}} F^n(U) \quad U = [L, F(L)]. \text{ interval}$$

in $\mathcal{L}_S =$ leaf space $\tilde{W}^cs$. Then either

1) $V = \tilde{M}$ then $\tilde{W}^cs$ is $\mathbb{R}$ -covered

(meaning $\mathcal{L}_S \cong \mathbb{R}$) and uniform :=

for any $L, E \in \tilde{W}^cs$ $d_{Haus}(L, E) < \infty$.

- u) $V \neq \tilde{M}$, each leaf in ∂V is fixed by f
 and ∂V projects to an f -invariant proper
 submanifold of W^s .

Proof i) $V = \tilde{M} \Rightarrow W^s \subset \mathbb{R}$ -convex.
 for any $E, F \in W^s$ $E \in [f^L(L), f^{L+1}(L)]$
 $F \in [f^0(L), f^{0+1}(L)]$.

$d_{\text{Haus}}(L, f(L)) \leq k$ is finite

$d(f(x), x) < k \quad \forall x$.

$d_{\text{Haus}}(f^L(L), f^0(L))$ finite

$\downarrow E$
 separates
 if not
 one of the
 endpoints.

u) $V \neq \tilde{M}$

$z \in \partial V$ leaf

$\nexists f(z) \neq z \quad X \quad V$

V is precisely invariant



$\nexists \gamma \in \pi_1(M) \quad \gamma(V) \cap V \neq \emptyset$ then $\gamma(V) = V$
 if not some boundary leaf of V is mapped
 inside of V — not separated from other
 leaves — not true in V

∂V is also precisely invariant

∂V projects to a submanifold of W^s in M . □

goal - Proposition - If M is hyperbolic

or selfst then option (U) of the above proposition cannot happen.

There is a technical problem to prove this.

Def A sublamination Λ of W^s is f -minimal if Λ is f -invariant and minimal w.r.t to this property.

Lemma let $\tilde{f}$ be a good lift of f to $\tilde{M}$.

Suppose Λ is an empty f -minimal sublamination of W^s_f so that every leaf of $\tilde{\Lambda} = \tilde{\pi}(\Lambda)$ is fixed by $\tilde{f}$. Then $\tilde{f}$ has no fixed points in $\tilde{\Lambda}$.

(this applies to our situation — if $L \in W^s$ is fixed by f then so is any ded translate $\sigma(L)$ — because $\tilde{f}$ commutes with $\tilde{\pi}_f(M)$. $\tilde{f} \sigma(L) = \sigma \tilde{f}(L) = \sigma(L)$.)

Proof. Very easy in the non branching situation

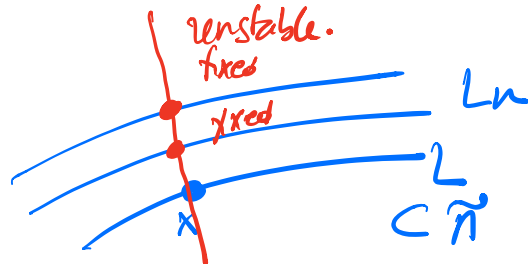
Suppose $x \in L \in \tilde{\Lambda}$ is fixed by $\tilde{f}$.

Under our hypothesis

L is not isolated in $\tilde{\Lambda}$

$L_n \in \tilde{\Lambda} \quad L_n \rightarrow L$

$\tilde{f}(L) = L$



$$\tilde{f}(u(x)) = u(x).$$

$u(x)$ is transverse to $\tilde{W}^s \Rightarrow$ intersects
a leaf at most once. single points

$$\tilde{f}(u(x) \cap L_n) = \tilde{f}(u(x)) \cap \tilde{f}(L_n) = u(x) \cap L_n$$

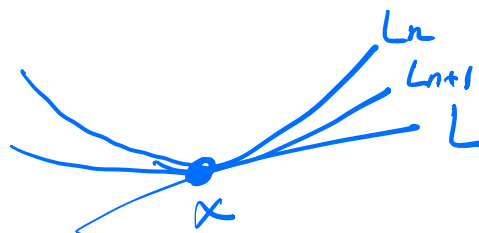
$\therefore$ only many fixed points in $u(x)$ X
of $\tilde{f}$ $\tilde{f}$ expands in $u(x)$.

This argument does not work when there is branching:

possible configuration

possibly all L_n

contain $x \rightarrow$



no contradiction in the above argument.

- Very complicated argument using bounded distance of $\tilde{f}$ which induces bounded distance in the leaves of $\tilde{\pi}$.

Consequences- $\tilde{f}$ acts freely on leaves of $\tilde{\pi}$.

Corollary of PH thm. id. $\tilde{f}$ good lift.

Λ non empty f -minimal subset of W^s
such that every leaf of $\tilde{\pi}$ is fixed by $\tilde{f}$.

Then every leaf in the minimal set

Λ is either a plane or a circle.

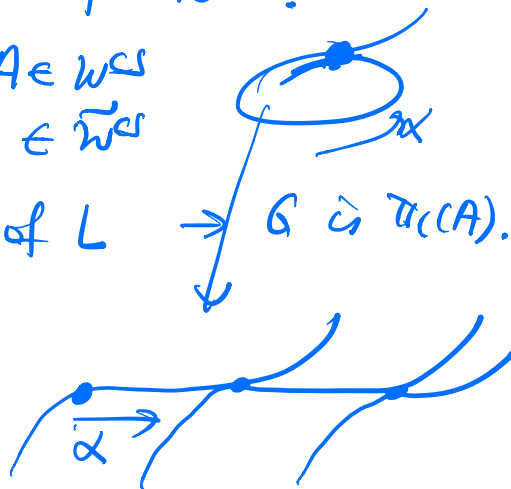
$\therefore$ (under orientation conditions).

✓ What is $\pi_1(M)$ of leaf of Wes ?

the fundamental group of $A \in Wes$ is the following - lift to $L \in \tilde{Wes}$

↓ let $G = \text{Isotropy group of } L \rightarrow G \subset \pi_1(A).$

equivalently $\downarrow$
in π_1 of the associated
leaf of approximating
foliation.



back to Lemma - show leaves are planes or annuli
Idea - show π_1 of such leaves A is at most $\mathbb{Z}$

No fixed points of $\tilde{f}$ in L a lift of A to $\tilde{M}$. $\tilde{f}$ acts freely.

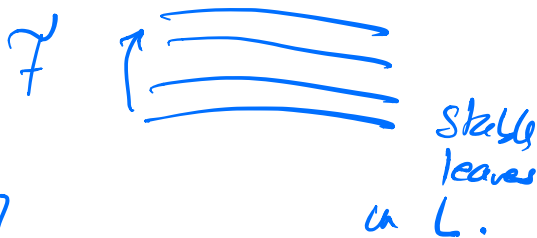
L has stable foliation, branching center foliation.

- A lot of analysis uses $\perp$ dim foliations in 2-dim sets. (properly embedded.)

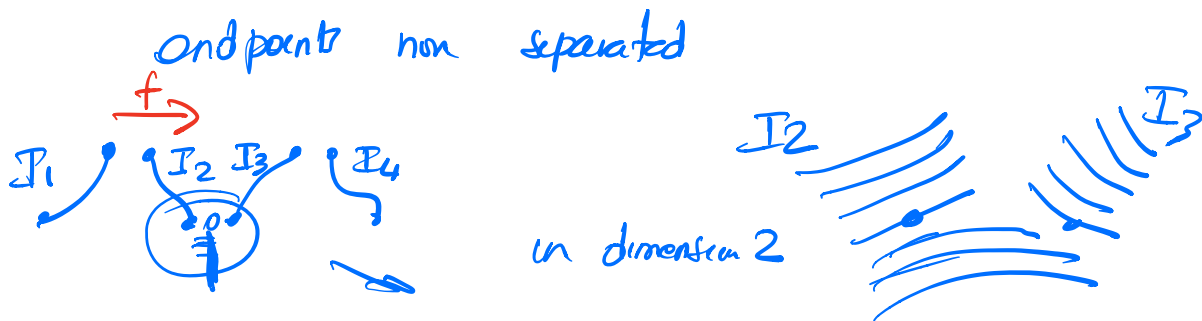
$\tilde{f}$ has an axis A in leaf space of stable foliation in L .

2 options

- either $A = \mathbb{R}$



- $A = \cup I_i = \cup [x_i, y_i]$
Segments



in other case
 if $\gamma \in \pi_1(A)$ γ acts freely in
 stable leaf space - else
closed stable leaf in $M \times X$

impossible f would contact more and more.

$\gamma, \tilde{f}$ act freely commute $\Rightarrow$ same axis.

if $\beta \in \pi_1(A)$ β also has same axis.

β, γ have same axis $\Rightarrow$ commute

either $\exists \mathbb{Z}^2 < \pi_1(A) \times \rightarrow$ torus
 or at most $\mathbb{Z}$ $\swarrow$ (or Klein bottle)
 compact

not possible $\pi_1(M)$ virtually solvable.

Prop M hyperbolic or Seifert. Optimal u)
 of proposition above cannot happen.

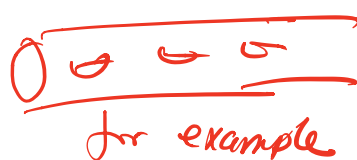
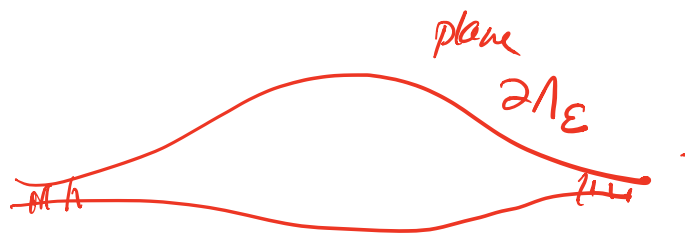
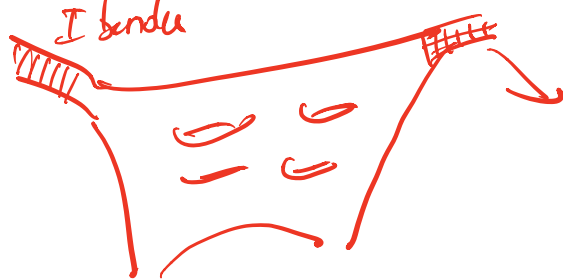
A submanifold coming from $2V$
 every leaf of X is fixed by $\tilde{f}$
 but leaves in V not fixed by $\tilde{f}$.

Here we can use approximating foliations.
 Approximating foliations W_ϵ^{cs} are actual foliations.
 The problem is that they are not f invariant.

Use W_ϵ^{cs} $\cap_\epsilon$ - complementary region -
 boundary leaves are planes or annuli
 of a plane

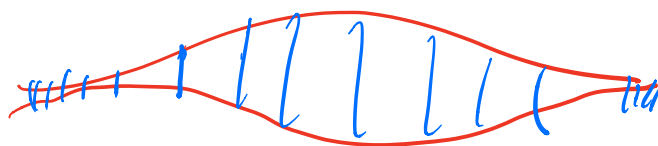
but complementary
 regions of a
 foliation - have

a core - thick part
 and arms - which are I -bundles
 I bundle



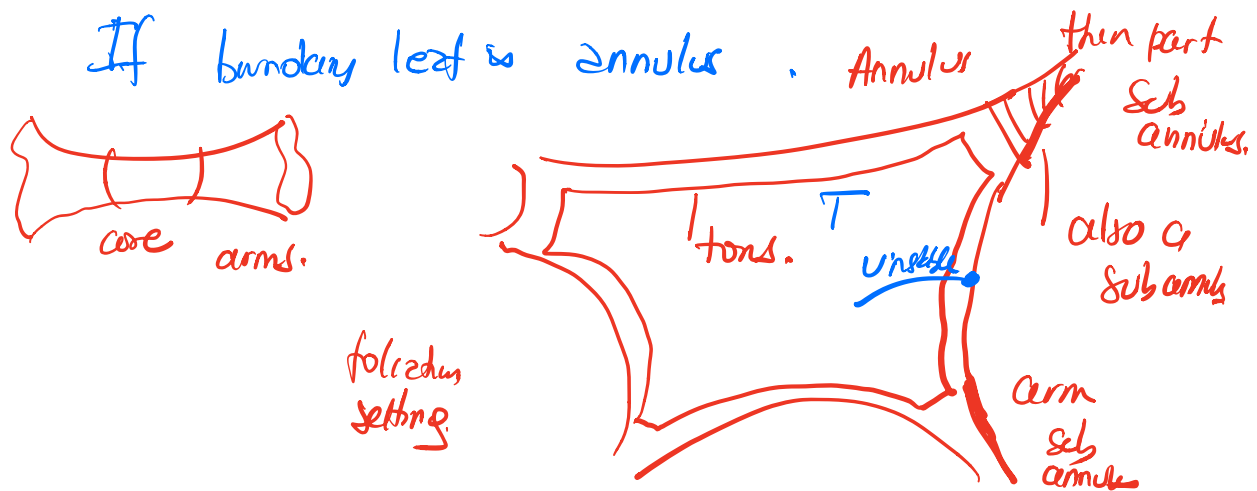
go back to W_ϵ^{cs} .

but all
 u unstable leaves
 unstable



middle is dark

iterate by f - unstable
 increase X .
 all unstable go through - maximum
 length unstable in between.



Now use M hyperbolic or Seifert

T either bounds a solid torus
 or contained in 3 ball X by
 project to W^S . leaves Π -invariant

use of iterate - length of unstable groups
 exponentially but volume of translates
 of fundamental region of solid torus

of groups linearly - contradiction

volume vs growth argument (Potrie + others) $\square$

Conclusion of M hyperbolic or Seifert

then option W cannot happen.

either T fixes every leaf of W^S or no leaf.

For M hyperbolic or Seifert 3 options

i) f fixes every leaf of $\tilde{W}^s$ and of $\tilde{W}^u$
Double invariant case.

ii) f fixes every leaf of $\tilde{W}^s$ but
no leaf of $\tilde{W}^u$
Mixed case.

iii) f fixes no leaf of either $\tilde{W}^s$ or $\tilde{W}^u$,
double transitive case.

Goal i) f is DAF ↓ more general.
Top
- Anosov flow in M
ii) not possible
iii) ~~the~~ for M hyp show $\exists$ top Anosov flow in M .

Analysis of i) assume f hom. to id., $\tilde{W}^s, \tilde{W}^u$
 f fixes every leaf of $\tilde{W}^s, \tilde{W}^u$.

Goal show f fixes every leaf of center foliation.

There are many steps -

Proposition

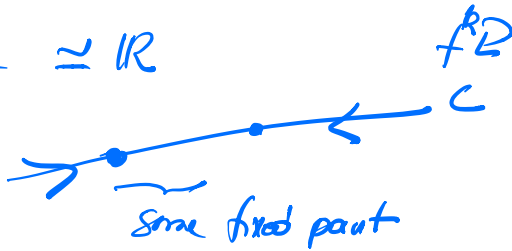
suppose f fixes every leaf of $\tilde{W}^s$ but no center leaf. Then every

periodic center leaf of f is coarsely contracting
and there are periodic center leaves.

loosely unbinding - leaf $\cong \mathbb{R}$

Idea $\rightarrow$

W^s has planes + annuli



If all planes

W^s all planes

Rosenberg

$$M = T^3 \quad X$$

$\pi(M)$ not solvable.

consider an annulus leaf A

with L a lift to $\tilde{M}$ γ generating $\text{isot}(L)$.

no $\tilde{f}$ fixed points.

$\tilde{f}$ there is an axis in stable foliation

suppose axis is γ .

figure in L

easy to see

$$\langle \gamma, \tilde{f} \rangle = \mathbb{Z}$$

does not act

freely

$$\text{acts on } \cup I_i = \mathbb{Z}$$

there are $n, L \neq 0$

$\gamma^n \tilde{f}^i$ has a fixed leaf.

$$L > 0$$

$$\gamma^n \tilde{f}^i(s) = s$$

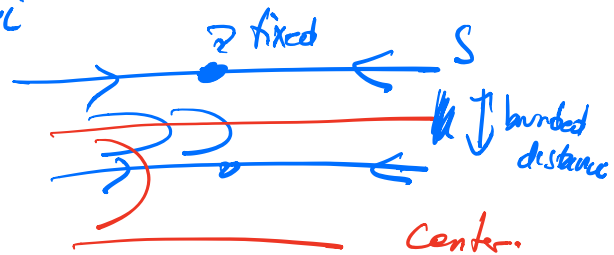
fixed point

γ cohomology

$\tilde{f}$ contracts stable



$\tilde{f}^i$ fixes a center - periodic



Contract

center foliation

$S, \tilde{f}(S)$ are a bounded distance in L .

$\tilde{f}$ moves points a bdd distance in any $E \in \tilde{W}^s$

$\tilde{f}(E) = E$.

So you have a center periodic and contracting.

This always happens - if center is periodic

there are n, i ~~so~~ $\beta^n \tilde{f}^i(c) = c$

$\tilde{f}$ preserves cs leaf E containing c .

hence β^n also does $\pi(E)$ is annulus.

Corollary. Double invariance case. If $\tilde{f}$ fixes every cs leaf, every cu leaf then
 $\hat{=}$ center leaf fixed by $\tilde{f}$.

- Suppose not then some $\tilde{f}$ fixes every $\tilde{W}^s$ but no $\tilde{W}^{cu}$
 previous lemma shows every periodic center leaf^{center} is coarsely contracting and there are some
 Now apply same result to $\tilde{f}$ fixes $\tilde{W}^{cu}$ leaf
 every periodic center is coarsely expanding.
 impossible.

Proposition If $\tilde{f}$ fixes every leaf of $\tilde{W}^s$, $\tilde{W}^{cu}$ then $\tilde{f}$ fixes every center leaf.

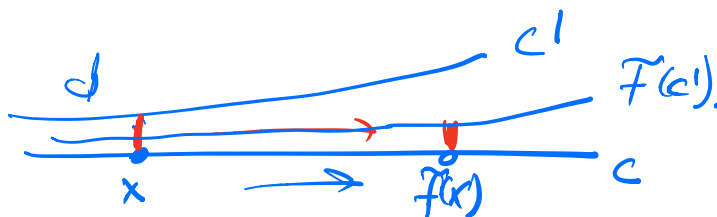
Idea Set of fixed centers is open.

obviously $\pi_1 M$ invariant because $\tilde{f}$ commutes with $\pi_1 M$

let c $f(c)=c$ let L be WS leaf containing c . $f(L)=L$.

Near c in L any c' center leaf is fixed by f .

no f fixed part



c' nearby

then $f(c')$, c' intersect a common unstable leaf

& $c' \neq f(c') \rightarrow$ contradiction as follows

c' , $f(c')$ are in the same cu leaf U .

U intersects a transversal stable segment twice - either $c' = f(c')$

or at some point they diverge - intersect stable segment.

& not same



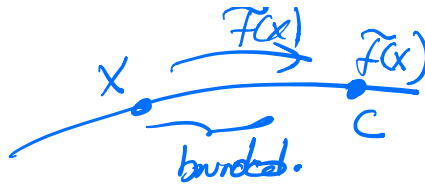
impossible since stable segment can intersect WS leaf at most once. $\square$

Dynamical coherence + DAF.

now we know f fixes every center leaf.

Lemma $f: M \rightarrow M$ PH hom. to Id. f -minimal WS, WCU. Let f be a good lft fixing every center leaf. Then there is $\geq$ global bound on the lengths between x and $f(x)$

in its center leaf.

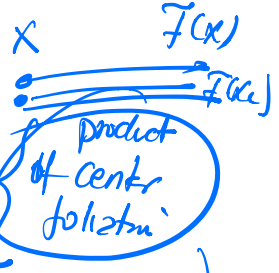


Ideas for non branching foliation.

if not bounded let $x_i, d_i(x_i, F(x_i)) \rightarrow \infty$.

$d(x_i, F(x_i))$ is bounded.

up to $\mathbb{N}(M)$ subsequence $x_i \rightarrow x$ & $x_i \rightarrow F(x_i) \rightarrow F(x)$.



$\Rightarrow$ lengths $x_i \rightarrow F(x_i)$ are bounded.

branching foliation - do the analysis not in $\tilde{M}$ directly, but in the center leaf space 2-dim (maybe non Hausdorff).

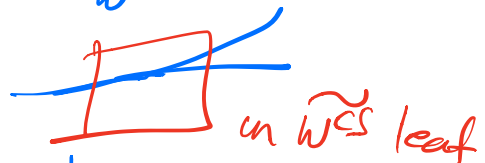


product
nbd in
the center leaf
space.

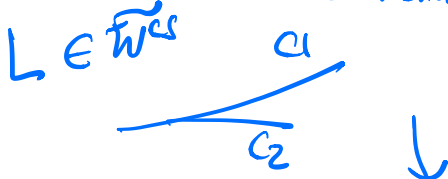
Corollary

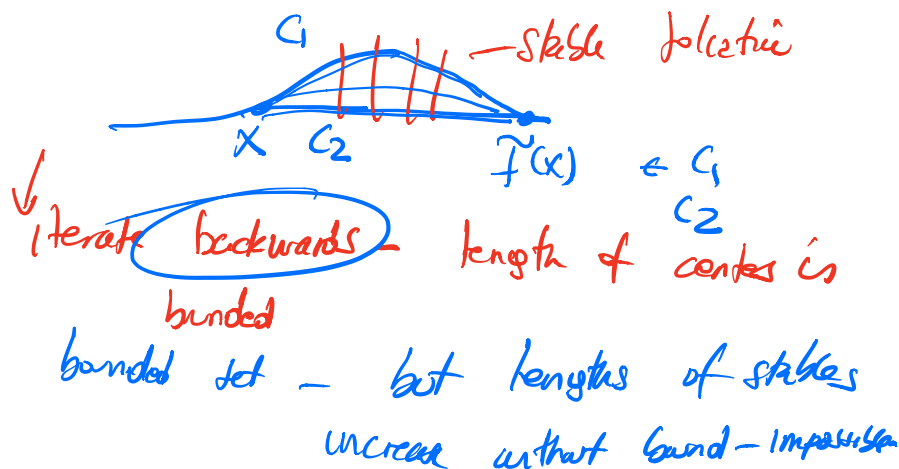
$f: M \rightarrow M$ homo. wd, f -minimal wcs, wcu. If some good lift fixes cusp center leaf then f is DC.

Pf suppose $\exists$ branching wcu



See branching in wcs leaf.





NO BRANCHING.

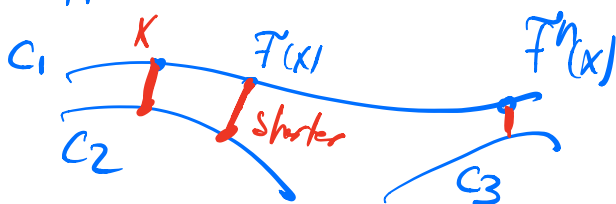
$\Rightarrow f$ is DC W^S, W^{cu} foliations.

Corollary f is DAF (in particular of H)

in $\geq$ cs leaf

center foliation is Hausdorff

Suppose not

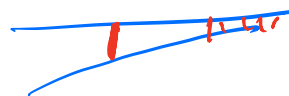


C_2, C_3 are separated.

C_1 very close

Iterate x eventually near C_3
but stakes decrease

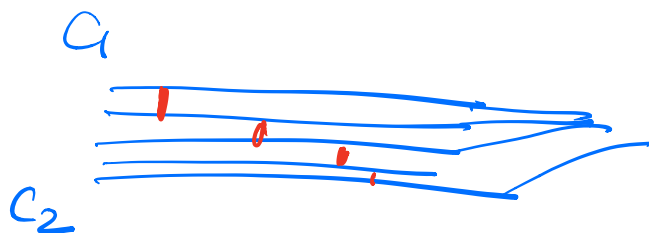
Impossible.



If C_1, C_2 intersect stable segment $\Rightarrow$
asymptotic in the f direction

Any 2 centers asymptotic in $\mathbb{F}$ direction

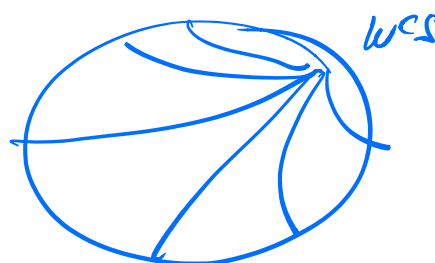
interpolate $q \rightarrow q_2$
 using connecting
 leaves intersect
 a common
 stable.



C_1 asymptotic to C_2

backwards they diverge.

So ^{center foliation} this is the
 flow foliation of a
 topological Anosov flow.



(weak) stable = w^s
 unstable = w^u .